POINT SETS AND ALLIED CREMONA GROUPS*

(PART II)

BY

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Introduction

Part I‡ of this account was devoted to a study of the invariants of a set P_n^k of n points in S_k under the group of permutations of the points. The set as a projective figure was mapped by a point P of a space $\Sigma_{k(n-k-2)}$ in which the permutation group appeared as a Cremona group $G_{n!}$. In this part we shall consider the effect upon the set P_n^k of certain Cremona transformations C in S_k. These transformations C are of a special charatcer when $k > 2$ described hereafter by the term *regular*. They are determined essentially by their fundamental points alone and in all important particulars are entirely analogous to the ternary Cremona transformations. These regular Cremona transformations form the *regular Cremona group in* S_k. If one set of fundamental points of a regular transformation C be placed at P_n^k there is determined a new set $P_n'^k$ *congruent to* P_n^k *under* C. The totality of sets $P_n'^k$ congruent in some order to a given set P_n^k is mapped in $\Sigma_{k(n-k-2)}$ by an aggregate of points P' which form a conjugate set under the *extended group* $G_{n,k}$ of P_n^k. This group $G_{n,k}$ in Σ contains $G_{n!}$ as a subgroup and in general is infinite and discontinuous. The major part of this article is devoted to a study of this group.

In § 5 a group $g_{n,k}$ of linear transformations which is isomorphic with $G_{n,k}$ is introduced. The new group brings to light properties both of $G_{n,k}$ and of regular transformations in S_k. An interesting result is a determination of all types of regular transformations with a single symmetrical set of fundamental points. Most of the types are well known but some are novel. In § 6 another group $e_{n,k}$ of linear transformations, also isomorphic with $G_{n,k}$, is defined, which is particularly effective for a discussion of the infinite groups $G_{9,2}$, $G_{8,3}$, and $G_{9,5}$.

The close relation between the associated sets P_n^k and Q_n^{n-k-2} which appeared

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‡ These Transactions, vol. 16 (1915), p. 155.

throughout Part I is here maintained. These sets have extended groups which are identical while their groups g and e are conjugate under linear transformation.

The first three paragraphs are devoted to the sets P_n^2 partly because the transformations C in S_2 are better known and partly because all the finite types of $G_{n,k}$, viz.: $G_{6,2}$, $G_{7,2} = G_{7,3}$, and $G_{8,2} = G_{8,4}$, occur here. In § 1 a particular order of congruence in S_2 is defined which is applied in § 2 to set forth conditions for congruence in S_2. In § 3 the extended group $G_{n,2}$ is defined and discussed, and the finite cases are identified with important known groups. In § 4 the extended group $G_{n,k}$ is considered.

The invariants of $G_{n,k}$ are taken up in § 8. Only two cases are treated—the $G_{9,2}$ as a sample of the infinite type and the $G_{7,2}$ as a sample of the finite type. The latter case presents a method of attacking the invariants of the ternary quartic which may turn out to be of value.

In this paper no applications of $G_{n,k}$ to the theory of equations are made such as appear for $G_{n!}$ at the end of Part I. The possibilities in this direction will be clear from a later article in which there is presented the important rôle of the extended group $G_{6,2}$ of order 51840 in the problem of determining the lines of a cubic surface.*

* It is not the purpose of this article to develop specific facts concerning definite types of transformations C in S_2. For these, treatises such as Clebsch-Lindemann, *Leçons sur la géométrie*, vol. II, sec. I, chap. IX; Doehlemann, *Geometrische Transformationen*, part II; and Sturm, *Die Lehre von den geometrischen Verwandschaften*, part IV, are available. So far as the author is aware the following points concerning C in S_2 are novel: (a) the definition and use in §§1, 2 of a definite mutual order of the F-points of C^{-1} and C; (b) the proof in §3 that, for a general point set, congruence does not imply projectivity except in the few particular cases enumerated in §2; (c) the development of the group $G_{n,2}$ in §3, of its algebraic relation to the set P_n^2, and of the group $e_{n,2}$ in §6; (d) the association of C in S_2 with regular transformations in hyperspace; (e) the determination in §6 of all types of C in S_2 with 9 or fewer F-points; and (f) the invariants of P_7^2 and P_9^2 under $G_{7,2}$ and $G_{9,2}$ respectively (§8).

In order to present these matters adequately it has been necessary to use, and occasionally convenient to re-prove, some known facts whose origin it is not easy to ascertain. The most original and comprehensive advance in this field is due to S. Kantor in the attempt to determine all types of finite Cremona groups. This work has been perfected by A. Wiman, *Zur Theorie der endlichen Gruppen* . . . , Mathematische Annalen, vol. 48 (1897), p. 195, where full references to the articles of Kantor and others are given. The mappings in §4 from the plane of P_6^2, P_7^2, and P_8^2 to respectively the cubic surface, the quartic curve, and the space sextic of genus four on a quadric cone date back to Clebsch and Noether and constantly reappear in later papers (cf. Wiman, loc. cit.). Articles along these general lines have been published by Snyder, these Transactions, vol. 11 (1910), p. 371 and vol. 12 (1911), p. 354; American Journal of Mathematics, vol. 33 (1911), p. 327.

The notion of a regular transformation in spaces of three or more dimensions appears to be new, as well as the entire discussion of $G_{n,k}$, $g_{n,k}$, and $e_{n,k}$ ($k > 2$) which is based on this notion. But particular regular transformations occur frequently in the literature. Thus for the Geyser involution determined by P_6^3 cf. Sturm, loc. cit.; and for the involution of order 15 determined by P_7^3 cf. Conner, American Journal of Mathematics, vol. 38 (1916).

1. Congruence of point sets in S_2

The general Cremona transformation* C_m of *order* m from the plane E_x to the plane E_y is established by defining a projective correspondence between the lines of E_y and the curves of order m of a net in E_x such that two curves of the net which correspond to lines on a point y have a single variable intersection at x. The remaining $m^2 - 1$ fixed intersections of the two curves arise by requiring that all the curves of the net pass through ρ *fundamental points* or *F-points*, $p_1, \cdots, p_\rho$, with a multiplicity r_i at the point $p_i (i = 1, \cdots, \rho)$. The inverse transformation C_m^{-1} of the same order from E_y to E_x determines a similar net of curves on E_y which pass through the same number ρ of F-points on E_y, $q_1, \cdots, q_\rho$, with the multiplicity s_j at the point q_j $(j = 1, \cdots, \rho)$. The F-points in either plane have no definite correspondents in the other. The directions about the F-point p_i on E_x correspond to the points of a *fundamental curve* or *F-curve* $F_i(y)$ on E_y which is rational, of order r_i, and completely determined by the number α_{ij} of times it passes through the F-points q_j on E_y. From this definition α_{ij} is the number of directions at p_i which correspond to directions at q_j whence α_{ij} is also the number of times the F-curve $F_j(x)$ on E_x which corresponds to q_j on E_y passes through the point p_i. We shall be concerned with the points p_i and q_j only as they may happen to form a part of a general point set P_n^2 or Q_n^2 and therefore will assume hereafter that they are in general position.

By proper conventions it is possible to order the two sets of F-points p, q with respect to each other. Let $p_{a_1}, \cdots, p_{a_g}$ be those points of the set p which have the same multiplicity r; $q_{b_1}, \cdots, q_{b_h}$ those points of the set q which have the same multiplicity s. Then the elements of the matrix $\|\alpha_{ij}\|$ $(i = a_1, \cdots, a_g; j = b_1, \cdots, b_h)$ in general are all alike. There will however be just one particular group of the q's for which $g = h$ and for which also the elements of the square matrix $\|\alpha_{ij}\|$ all have the same value β except for a single element in each line which has the value $\gamma \neq \beta$.† If then for a particular order of $p_{a_1}, \cdots, p_{a_g}$ we arrange the points $q_{b_1}, \cdots, q_{b_g}$ so that the elements γ fall along the principal diagonal of the square matrix a point of the group p corresponds to a definite point of the group q and this correspondence is independent of the particular order in the group p. This convention fails if $g = 2$ or if $g = 1$. If $g = 2$ we require in addition that the elements γ in the principal diagonal be greater than the elements β in the other diagonal. The number σ of groups p for which $g = 1$ is the same as the number of groups q for which $g = 1$. If $p_1, \cdots, p_\sigma$ be the points in these σ groups arranged in order of decreasing multiplicity and if $q_1, \cdots, q_\sigma$ be the similar points similarly

* The properties of this transformation as set forth in Clebsch-Lindemann (loc. cit.) are assumed here.

† Cf. Clebsch-Lindemann, loc. cit., pp. 202–5.

arranged we shall say that p_e and q_e $(e = 1, \cdots, \sigma)$ correspond. By means of the conventions just adopted there is established a definite correspondence between the two sets of F-points of C_m.

We shall say that the point sets P_n^2 and Q_n^2 $(n \geqq 5)$ are *congruent under the Cremona transformation* C_m with ρ F-points if ρ of the pairs p_i, q_i $(i = 1, \cdots, n)$ are corresponding F-points of C_m as defined above and if the remaining $n - \rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. Here the requirement $n \geqq \rho$ is an essential part of the definition while the requirement $n \geqq 5$ merely bars the cases $n = 3, 4$ where the definition might apply for $m = 2$ but would not imply a projective property of the point sets. The definition also requires that none of the $n - \rho$ points p or $n - \rho$ points q shall lie on the F-curves of C_m which is of course in line with the hypothesis that P_n^2 or Q_n^2 is a general set.

(1) *The general Cremona transformation C_m $(m > 2)$ with ρ F-points is projectively determined when there is given the order m, the ρ F-points in E_x, their multiplicities subject to the conditions $\sum_{i=1}^{i=\rho} r_i^2 = m^2 - 1$, $\sum_{i=1}^{i=\rho} r_i = 3(m-1)$, and the positions in E_y of four corresponding F-points.*

The multiplicities of the F-points on E_x determine the net. The F-curves on E_x are determined by the requirement that they have no variable intersection with curves of the net. The orders of the F-curves determine the multiplicities s_j and the numbers α_{ij} so that the particular four F-curves which correspond to the four given points on E_y can be picked out. The residue of each of the four F-curves with regard to the net is a definite pencil of curves so that the four pencils of curves of the net on E_x corresponding to the four line pencils on the given points of E_y are determined. Hence the projective correspondence between the net of curves on E_x and the net of lines on E_y is known and C_m is determined.

(2) *The projective conditions for the congruence of two sets $P_{\rho+1}^2$ and $Q_{\rho+1}^2$ under C_m $(m > 2)$ with ρ F-points imply the projective conditions for the congruence of the sets P_n^2 and Q_n^2 $(n > \rho + 1)$ under C_m.*

For among the pairs of $P_{\rho+1}^2$ and $Q_{\rho+1}^2$ there occur the ρ pairs of corresponding F-points of C_m. If the data of theorem (1) are given and if the further conditions which determine from these data the location of the remaining $\rho - 4$ F-points on E_y have been obtained in some fairly convenient form,* then the conditions for the occurrence of the $(\rho + 1)$-th pair amount to the projective construction of a pair of ordinary corresponding points. This construction can then be applied to the remaining $n - (\rho + 1)$ corresponding pairs. The above remarks apply also to the particular case $m = 2$ if $\rho + 1$ be replaced by $\rho + 2$.

* The form suggested in Clebsch-Lindemann, loc. cit., p. 203, footnote (2), would seldom be practically useful.

It is clear that some sort of expression of the conditions for congruence is indispensable for the utilization in geometry or analysis of the Cremona transformations of higher degree. In § 2 projective statements of these conditions for the cases $n < 9$ are given. In § 3 there is indicated a method for stating these conditions analytically which applies to any case.

If the sets P_n^2 and Q_n^2 are congruent under C_m, the first ρ pairs being F-points of C_m, the arithmetical effect of C_m upon curves in E_x which are related to the set P_n is expressed by a linear transformation $L(C_m)$* which for convenience is given here. Let a curve in E_x of order z_0 have the multiplicity z_i at the point p_i of P_n^2. It is transformed by C_m into a curve in E_y of order z_0' which has the multiplicity z_j' at the point q_j of Q_n^2. Then

$$
(3) \qquad L(C_m):\quad \begin{aligned} z_0' &= m z_0 - \textstyle\sum_{i=1}^{i=\rho} r_i z_i, \\ z_j' &= s_j z_0 - \textstyle\sum_{i=1}^{i=\rho} \alpha_{ij} z_i, \\ z_l' &= \qquad -(-1) z_l, \end{aligned} \quad \begin{pmatrix} j = 1, \cdots, \rho; \\ l = \rho + 1, \cdots, n \end{pmatrix}.
$$

It is worth noting that when the z's and z''s refer to the same coördinate system $L(C_m)$ is unique only by virtue of our conventions in regard to the order of congruence. If these were disregarded L might be any one of $n!$ such transformations.

2. Projective conditions for congruence in S_2

In this paragraph the projective conditions on the two sets of ρ F-points of C_m, and the further conditions on a pair of ordinary corresponding points when $n > \rho$, will be determined for all transformations C_m for which $n < 9$. This is a natural limit since the number of types of C_m is infinite for $\rho \geqq 9$. The conditions are given in the form of constructions for the points to be determined, all of which are of rather elementary character except possibly that for the ninth base point of a pencil of cubics.†

The various types of C_m to be considered are furnished by the following table where α_j is the number of F-points of multiplicity j:

	C_2	C_3	C_4	D_4	C_5	D_5	D_6	E_6	E_6^{-1}	D_7	E_7	D_8	E_8	E_8^{-1}	E_8'	E_9	E_9^{-1}	E_9'
α_1	3	4	3	6	0	3	1	4	3	0	2	0	2	0	1	1	0	0
α_2		1	3	0	6	3	4	1	4	3	3	0	0	5	3	1	3	4
α_3				1		1	2	3	0	4	2	7	5	2	2	3	3	0
α_4								–	1		1		1	0	2	3	1	4
α_5														1			1	

* This transformation has been employed by S. Kantor in his crowned memoir of 1883: *Theorie des transformations periodiques univoques:* Naples (De Rubertis), 1891, p. 293.

† Cf. Clebsch-Lindemann, loc. cit., vol. III, p. 451.

	E_{10}	E_{10}^{-1}	E'_{10}	E''_{10}	E_{11}	E'_{11}	E_{12}	E_{12}^{-1}	E'_{12}	E_{13}	E'_{13}	E_{14}	E'_{14}	E_{15}	E_{16}	E_{17}
α_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
α_2	0	1	2	0	1	0	1	0	0	0	0	0	0	0	0	0
α_3	2	5	2	7	2	4	0	3	2	1	0	1	0	0	0	0
α_4	5	0	3	0	3	3	4	1	4	3	6	0	3	1	0	0
α_5		2	1	0	2	0	3	4	1	3	0	6	3	4	3	0
α_6				1		1			1	1	2	1	2	3	5	8

That the table* is complete and that each type exists can be verified by taking the products of the various types and a quadratic transformation whose three F-points are placed either at F-points of C_m or at F-points and ordinary points if thereby the product has not more than 8 F-points. The notation for the transformations is so chosen that the letter C, D, E indicates 6 or fewer, 7, 8 F-points respectively; the subscript indicates the order; and the superscript indicates an arrangement according to the maximum multiplicity of the F-points. The well-known types C_2, C_5, D_8, and E_{17} contain the F-points symmetrically and will prove especially useful.

In what follows the first of the two subscripts of a point indicates its order in a set and the second indicates its multiplicity as an F-point of C_m. We shall now take up the particular types.

C_2: This transformation requires two congruent sets of four points for its determination. Two sets of five points, $p_{1,1}$, $p_{2,1}$, $p_{3,1}$, $p_{4,0}$, $p_{5,0}$ and $q_{1,1}$, $q_{2,1}$, $q_{3,1}$, $q_{4,0}$, $q_{5,0}$ are congruent under C_2 when *the set P_5^2 as wri ten is projective to the set Q_5^2 with the last two points interchanged.* We can infer from this fact that the general transformation C_m can be constructed by projectivities alone. The fact can be verified analytically and extended easily to the transformation $C^{(k)}$ in S_k obtained by inverting the variables and then effecting a projectivity.

C_5: Two sets of seven points $p_{1,2}, \cdots, p_{6,2}, p_{7,0}$ and $q_{1,2}, \cdots, q_{6,2}, q_{7,0}$ are congruent under C_5 when the sets $p_{1,2}, \cdots, p_{6,2}$ and $q_{1,2}, \cdots, q_{6,2}$ are the associated sets whose construction is given in I § 1 (10). Then the sextic line pencil from $p_{7,0}$ to P_6^2 is projective to that from $q_{7,0}$ to Q_6^2.

C_4: If the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $q_{1,1}, \cdots, q_{3,1}$, $q_{4,2}, \cdots, q_{6,2}, q_{7,0}$ are congruent under C_4, then the product of C_4 and a C_5 with congruent sets $q_{1,2}, \cdots, q_{6,2}, q_{7,0}$ and $r_{1,2}, \cdots, r_{6,2}, r_{7,0}$ is a C_2 with congruent sets $p_{1,0}, \cdots, p_{3,0}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $r_{1,0}, \cdots, r_{3,0}$, $r_{4,1}, \cdots, r_{6,1}, r_{7,0}$. Given then the set P_7^2 we construct the auxiliary set R_7^2 congruent to P_7^2 under a C_2 of the above type and then construct the required set Q_7^2 as congruent to R_7^2 under C_5.

* A similar table is given by Kantor, l. c., p. 280 which includes a non-existent type, $m = 11$, $\alpha_3 = 5$, $\alpha_5 = 3$.

C_3: If the F-points of C_3 are $p_{1,2}, p_{2,1}, \cdots, p_{5,1}$ and $q_{1,2}, q_{2,1}, \cdots, q_{5,1}$, the product of C_3 and the C_2 defined by the congruent sets $q_{1,1}, \cdots, q_{3,1}, q_{4\,0}, q_{5,0}$ and $r_{1,1}, \cdots, r_{3,1}, r_{4,0}, r_{5,0}$ is a C_2' for which $p_{1,1}, p_{2,0}, p_{3,0}, p_{4,1}, p_{5,1}$ are congruent to $r_{1,1}, r_{3,0}, r_{2,0}, r_{5,1}, r_{4,1}$. This requires *that* $p_1, \cdots, p_5$ *and* $q_1, \cdots, q_5$ *be congruent under a projectivity* π. Since both π and C_3 transform the line $\overline{p_1, p_i}$ into $\overline{q_1, q_i}$ and the conic $\overline{p_i\, p_j}\,\overline{p_k\, p_l}$ into $\overline{q_i\, q_j}\,\overline{q_k\, q_l}$ $(i, j, k, l = 2, 3, 4, 5)$ their effect on the line pencil on p_1 and on the conic pencil on $p_2, \cdots, p_5$ is the same. If then π transforms $p_{6,0}$ into $\overline{q}_{6,0}$ the correspondent $q_{6,0}$ of $p_{6,0}$ under C_3 is the partner of $\overline{q}_{6,0}$ in the involution cut out on the line $q_{1,2}\,\overline{q}_{6,0}$ by conics on $q_2, \cdots, q_5$.

D_8: Let the F-points of D_8 be $p_{1,3}, \cdots, p_{7,3}$ and $q_{1,3}, \cdots, q_{7,3}$. Let F_{p_i} and F_{q_i} be the F-curves with double points at p_i and q_i respectively. Since the pencil of cubics with a given direction at q_1 and on Q_7^2 meets F_{q_1} in a definite point there is a binary projectivity between directions at q_1 and points on the rational cubic F_{q_1} in which the directions on $F_{q_2}, \cdots, F_{q_7}$ correspond to the points $q_2, \cdots, q_7$. By cutting F_{q_1} by lines on q_1 we find that the points $q_2, \cdots, q_7$ on F_{q_1} are projective to the line pencil $q_1 - q_2, \cdots, q_7$. Hence these six lines are projective to the tangents of $F_{q_2}, \cdots, F_{q_7}$ at q_1. Since lines on p_1 are projective under D_8 to the quintic pencil $q_1, q_2^2, \cdots, q_7^2$ or to its tangent line pencil at q_1 we see that the six lines $p_1 - p_2, \cdots, p_7$ are projective to the tangents at q_1 of $F_{q_2}, \cdots, F_{q_7}$ and therefore to the six lines $q_1 - q_2, \cdots, q_7$. Hence *the sets* P_7^2 *and* Q_7^2 *congruent under* D_8 *are also congruent under a projectivity* π. But D_8 and π have the same effect upon the 21 degenerate cubics of the net on P_7^2 and therefore upon every curve of the net as well. If then $p_{8,0}$ and $q_{8,0}$ correspond under D_8 and if $p_{8,0}$ and $\overline{q}_{8,0}$ correspond under π, $q_{8,0}$ is the ninth base point of the pencil of cubics on Q_7^2 and $\overline{q}_{8,0}$.

In the case of C_4 we used the fact that C_4 and C_2 are paired by multiplication with C_5. We can now use the pairing of D_4, D_5, D_6, D_7 with C_5, C_4, C_3, C_2 respectively by multiplication with D_8 in order to obtain the conditions for congruence under these new transformations. This process is easily followed out and only the results are given below.

D_4: If the sets $p_{1,3}, p_{2,1}, \cdots, p_{7,1}, p_{8,0}$ and $q_{1,3}, q_{2,1}, \cdots, q_{7,1}, q_{8,0}$ are congruent under D_4 then the sets $p_2, \cdots, p_7$ and $q_2, \cdots, q_7$ are associated and $p_{1,3}, q_{1,3}$ correspond under the C_5 determined by the associated sets. If also $p_{8,0}, \overline{q}_{8,0}$ correspond under this C_5 then $q_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_5: If the sets $p_{1,1}, \cdots, p_{3,1}, p_{4,2}, \cdots, p_{6,2}, p_{7,3}, p_{8,0}$ and $q_{1,2}, \cdots, q_{3,2}, q_{4,1}, \cdots, q_{6,1}, q_{7,3}, q_{8,0}$ are congruent under D_5 then the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, \cdots, p_{6,1}, p_{7,0}$ and $q_{1,1}, \cdots, q_{3,1}, q_{4,2}, \cdots, q_{6,2}, q_{0,7}$ are congruent under C_4. If the further ordinary pair $p_{8,0}, \overline{q}_{8,0}$ correspond under this C_4 then $q_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_6: If the sets $p_{1,1}, p_{2,2}, \cdots, p_{5,2}, p_{6,3}, p_{7,3}, p_{8,0}$ and $q_{1,1}, q_{2,2}, \cdots, q_{5,2}, q_{6,3}, q_{7,3}, q_{8,0}$ are congruent under D_6 then the sets $p_{1,2}, p_{2,1}, \cdots, p_{5,1}, p_{6,0}, p_{7,0}, p_{8,0}$ and $q_{1,2}, q_{2,1}, \cdots, q_{5,1}, q_{6,0}, q_{7,0}, \overline{q}_{8,0}$ are congruent under C_3 and $\overline{q}_{8,0}$ is the ninth base point of the pencil on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

D_7: If the sets $p_{1,2}, \cdots, p_{3,2}, p_{4,2}, \cdots, p_{7,3}, p_{8,0}$ and $q_{1,2}, \cdots, q_{3,2}, q_{4,3}, \cdots, q_{7,3}, q_{8,0}$ are congruent under D_7 then the sets $p_{1,1}, \cdots, p_{3,1}, p_{4,0}, \cdots, p_{7,0}, p_{8,0}$ and $q_{1,1}, \cdots, q_{3,1}, q_{4,0}, \cdots, q_{7,0}, q_{8,0}$ are congruent under C_2 and $\overline{q}_{8,0}$ is the ninth base point of the pencil of cubics on $q_1, \cdots, q_7, \overline{q}_{8,0}$.

There remain only the types E with 8 F-points. Since we seek the conditions for congruence of sets with at most 8 points the construction for ordinary pairs need not be considered further. As we shall prove later the sets P_8^2 and Q_8^2 for the type E_{17} are projective. If then $EE_{17} = C_m$ the conditions for congruence of 8 points under E and C_m are the same. In this way E_{17} pairs the transformations as follows: E_{16}, C_2; E_{15}, C_3; E_{14}, D_4; E'_{14}, C_4; E_{13}, D_5; E'_{13}, C_5; E_{12} and E_{12}^{-1}, E_6 and E_6^{-1}; E'_{12}, D_6; E_{11}, E_7; E'_{11}, D_7; E_{10} and E_{10}^{-1}, E_8 and E_8^{-1}, E'_{10}, E'_8; E''_{10}; D_8; while E'_9, and E_9, E_9^{-1} are paired with themselves. Hence to complete the discussion we have yet to consider only the types E_6, E_6^{-1}, E_7, E_8, E_8^{-1}, E'_8, E_9, E_9^{-1}, E'_9, and E_{17}.

E_6, E_6^{-1}: Since the product $E_6 D_8$ for proper choice of the F-points o D_8 is D_6 we have the following construction for the two sets of F-points of E_6, E_6^{-1}. If in the sets P_8^2 and Q_8^2 congruent under D_6 as above we replace q_1 by $\overline{q_1}$, the ninth base point of the pencil of cubics on Q_8^2 we have sets $p_{1,2}, p_{2,1}, \cdots, p_{5,1}, p_{6,3}, \cdots, p_{8,3}$ and $q_{8,4}, q_{2,2}, \cdots, q_{5,2}, q_{6,1}, q_{7,1}, \overline{q}_{1,1}$ congruent in this order under E_6, E_6^{-1}.

E_7: The product $D_5 D_8$ for isolated $q_{6,1}$ in the sets congruent as above under D_5 is E_7. If then we replace q_6 by $\overline{q_6}$, the ninth base point of the pencil on Q_8^2 we have sets $p_{1,2}, \cdots, p_{3,2}, p_{4,1}, p_{5,1}, p_{6,4}, p_{7,3}, p_{8,3}$ and $q_{1,2}, \cdots, q_{3,2}, q_{5,3}, q_{4,3}, q_{8,4}, \overline{q}_{6,1}, q_{7,1}$ congruent in this order under E_7.

E'_8: This type can be obtained from the product $C_4 D_8$. If in the sets congruent as above under C_4 and amplified by the pair $p_{8,0}, q_{8,0}$ we replace q_1 by $\overline{q_1}$ the ninth base point of a pencil on $q_1, \cdots, q_8$ we obtain the sets $p_{1,1}, p_{2,4}, p_{3,4}, p_{4,2}, \cdots, p_{6,2}, p_{7,3}, p_{8,3}$ and $\overline{q}_{1,1}, q_{2,3}, q_{3,3}, q_{4,2}, \cdots, q_{6,2}, q_{7,4}, q_{8,4}$ congruent in this order under E'_8.

E_8, E_8^{-1}: From the product $D_4 D_8$ we find that if in the sets congruent as above under D_4 we replace q_2 by $\overline{q_2}$ the ninth base point of the pencil on Q_8^2 we obtain the set of F-points of E_8^{-1}, viz., $p_{1,3}, p_{2,5}, p_{3,2}, \cdots, p_{7,2}, p_{8,3}$, which is congruent to the set of F-points of E_8, viz., $\overline{q}_{2,1}, q_{8,4}, q_{3,3}, \cdots, q_{7,3}, q_{1,1}$.

E'_9: From a proper product $C_5 C'_5$ we find that, for the sets congruent as above under C_5 and amplified by a further pair $p_{8,0}, q_{8,0}$, if we construct a

set $\overline{q_1}_{,0}, \overline{q_2}_{,0}, \overline{q_3}_{,2}, \cdots, \overline{q_8}_{,2}$ congruent under C_5' to $q_{1,0}, q_{2,0}, q_{3,2}, \cdots, q_{8,2}$, then the sets $p_{1,2}, p_{2,2}, p_{3,4}, \cdots, p_{6,4}, p_{7,2}, p_{8,2}$ and $\overline{q_2}_{,2}, \overline{q_1}_{,2}, \overline{q_3}_{,4}, \cdots, \overline{q_6}_{,4}, \overline{q_8}_{,2}, \overline{q_7}_{,2}$ are congruent in this order under E_9'.

E_9, E_9^{-1}: If the sets $p_{1,1}, \cdots, p_{4,1}, p_{5,2}, p_{6,0}, \cdots, p_{8,0}$ and $q_{1,1}, \cdots, q_{4,1}, q_{5,2}, q_{6,0}, \cdots, q_{8,0}$ are congruent under C_3, and if $\overline{q_1}$ is the ninth base point of the pencil on Q_8^2 then the set $p_{1,5}, p_{2,2}, \cdots, p_{4,2}, p_{5,4}, p_{6,3}, \cdots, p_{8,3}$ of F-points of E_9^{-1} is congruent to the set $q_{5,2}, q_{2,3}, \cdots, q_{4,3}, \overline{q_1}_{,1}, q_{6,4}, \cdots, q_{8,4}$ of F-points of E_9.

E_{17}: The product of E_{10}'' with congruent sets $p_{1,6}, p_{2,3}, \cdots, p_{8,3}$ and $q_{1,6}, q_{2,3}, \cdots, q_{8,3}$ by a D_8 with congruent sets $q_{1,3}, \cdots, q_{7,3}, q_{8,0}$ and $r_{1,3}, \cdots, r_{7,3}, r_{8,0}$ is a $\overline{D}_8$ with congruent sets $p_{1,3}, p_{2,3}, \cdots, p_{7,3}, p_{8,0}$ and $r_{8,3}, r_{2,3}, \cdots, r_{7,3}, r_{1,0}$, whence the sets $p_2, \cdots, p_7$ and $q_2, \cdots, q_7$ are projective since each is projective to $r_2, \cdots, r_7$. The product of this E_{10}'' by an E_{17} with congruent sets $q_{1,6}, \cdots, q_{8,6}$ and $s_{1,6}, \cdots, s_{8,6}$ is another D_8 with congruent sets $p_{1,0}, p_{2,3}, \cdots, p_{7,3}, p_{8,3}$ and $s_{1,0}, s_{2,3}, \cdots, s_{7,3}, s_{8,3}$. Hence $s_2, \cdots, s_7$ are projective to $p_2, \cdots, p_7$ and therefore to $q_2, \cdots, q_7$. These being any sets of six formed from the sets of 8 F-points of E_{17} there follows that *the sets P_8^2 and Q_8^2 congruent under E_{17} are projective.*

Of all the cases investigated above the only ones for which congruence implies projectivity are (1) that of 5 points under C_2 and C_3, (2) that of 7 points under D_8, and (3) that of 8 points under E_{17}. The order of congruence and of projectivity is the same in all these cases except that of 5 points under C_2. It is important to note however that the projectivity no longer exists when the congruence is extended to include additional pairs.

3. The extended group $G_{n,2}$

In § 1 we have defined the congruence of sets P_n^2 and Q_n^2 under the Cremona transformation C_m. According to (1) if the type of C_m and the location of its F-points among the points of P_n^2 are given, then the set Q_n^2 is projectively determined, i. e., if four of the points of Q_n^2 are fixed at a base the coördinates of the remaining points can be rationally expressed in terms of the coördinates of P_n^2. Of course the same fact is true if the sets be interchanged.

In I, § 6, we have mapped the sets P_n^2 upon the points P of a space $\Sigma_{2(n-4)}$. If we take the sets P_n^2 and Q_n^2 in the canonical form

$$P_n^2:\ \begin{array}{lll} 1, & 0, & 0, \\ 0, & 1, & 0, \\ 0, & 0, & 1, \\ 1, & 1, & 1, \\ x_i, & y_i, & u, \end{array} \quad (i = 1, \cdots, n-4); \qquad Q_n^2:\ \begin{array}{lll} 1, & 0, & 0, \\ 0, & 1, & 0, \\ 0, & 0, & 1, \\ 1, & 1, & 1, \\ x_j', & y_j', & u', \end{array} \quad (j = 1, \cdots, n-4);$$

then the coördinates of P and Q, the maps in $\Sigma_{2(n-4)}$ of P_n^2 and Q_n^2 respectively, are x_i, y_i, u and x_j', y_j', u'. Then the above statement can be expressed more briefly as follows:

(4) *If the sets P_n^2 and Q_n^2 are congruent under Cremona transformation their maps P and Q in $\Sigma_{2(n-4)}$ are corresponding points under a Cremona transformation in Σ.*

If P_n^2 is congruent in some order to Q_n^2 under C_m and Q_n^2 is congruent in some order to R_n^2 under C_μ then P_n^2 is congruent in some order to R_n^2 under the product $C_m \cdot C_\mu$ properly taken.* For the $\rho' \leqq n$ F-points of C_μ can all be placed at F-points of C_m^{-1} or at ordinary points of C_m^{-1} which correspond to points of P_n^2. Then the F-points of the product arise either from F-points of C_m or from ordinary points of C_m whose correspondents are F-points of C_μ and in either case are found in P_n^2. Similarly the F-points of the inverse product are found in R_n^2. If an F-curve of C_m is an F-curve of C_μ^{-1} some F-point of P_n^2 and some F-point of R_n^2 form an ordinary pair of the product. Ordinary pairs of both C_m and C_μ take care of themselves. Hence

(5) *Two point sets each congruent in some order to a third are congruent in some order to each other. The aggregate of sets Q_n^2 congruent in some order with a given set P_n^2 is mapped in $\Sigma_{2(n-4)}$ by an aggregate of points Q which form a conjugate set under the thus defined* EXTENDED GROUP *$G_{n,2}$ in $\Sigma_{2(n-4)}$ associated with the point set P_n^2.*

Since the general Cremona transformation in S_2 can be expressed as a properly arranged product of quadratic factors let us determine the effect upon the set P_n^2 given above of a quadratic transformation for which the first four points of P_n^2 are congruent to the first four points of Q_n^2. This transformation is $x' = 1/x$, $y' = 1/y$, $u' = 1/u$. Its effect upon the further points of P_n^2 is to invert their coördinates whence the corresponding transformation A *in* $\Sigma_{2(n-4)}$ is obtained by inverting the variables. Since any arrangement in a product can be secured by proper permutation of the points and any quadratic transformation can be obtained by applying such permutations to A we see that

(6) *The extended group $G_{n,2}$ of P_n^2 in $\Sigma_{2(n-4)}$ can be generated by the $G_{n!}$ of P_n^2 (whose generators are given in I, § 7 (64)) and the involutory transformation A which is obtained by inverting the variables in Σ.*

Thus for a general Cremona transformation C_m in S_2 with ρ F-points there will exist an operation of $G_{\rho,2}$ compounded from the above generators which will express the coördinates of the ρ F-points of C_m^{-1} rationally in terms of those of C_m and the same compound formed in $G_{\rho+1,2}$ will give the transformation itself. However the coördinate system in both planes is therewith specially selected. Conversely for every operation of $G_{n,2}$ there will exist for given

* The product is properly taken if the F-points of C_μ are included in the set Q_n^2.

P_n^2 a projectively defined set Q_n^2 which is in some order congruent to P_n^2 as defined in § 1.

The order of $G_{n,2}$ is the number of points Q conjugate to a general point P, i. e., the number of sets Q_n^2 congruent in any one of $n!$ arrangements with P_n^2 under any Cremona transformation with $\rho \leqq n$ F-points provided that no two congruent sets are projective in any order. This proviso is disposed of by the results of § 2 and by the theorem:

(7) *If a point set is congruent to a given* GENERAL *point set under Cremona transformation the two sets cannot be projective in any order if the number of points is nine or more.*

Let us prove that P_n^2, congruent to Q_n^2 under C_m, cannot be projective to Q_n^2 under a collineation K even in the special case when P_n^2 is on a cubic curve E^3 but otherwise unrestricted. We shall see in § 5 that when $\rho \geqq 9$ there exist no C_m's whose F-points all have the same multiplicity. Even when $\rho < 9$ and $n \geqq 9$ ordinary points of multiplicity zero occur so that when $n \geqq 9$ we can assume that points q_j and q_k occur in Q_n^2 whose multiplicities s_j and s_k under C_m are different. Let P_n^2 on E^3 have elliptic parameters $u_1, \cdots, u_n$ where $u + u' + u'' \equiv 0$ is the condition for three points of a line. Then C_m transforms E^3 into a cubic $E^{3'}$ on Q_n^2 in such a way that the point u of E^3 goes into the point u of $E^{3'}$. If K^{-1} transforms Q_n^2 into P_n^2 it must transform $E^{3'}$ back into E^3 so that the point u of $E^{3'}$ goes into the point u' of E^3. Since E^3 is thereby birationally transformed into itself, $u = \pm u' + b$. Now the points q_j and q_k as points u of $E^{3'}$ correspond to the meets outside P_n^2 of their F-curves with E^3, i. e., to $-\sum_{l=1}^{l=\rho} \alpha_{l,j} u_l$ and $-\sum_{l=1}^{l=\rho} \alpha_{l,k} u_l$; and as points of Q_n^2 they correspond under K^{-1} to points $p_{j'}$ and $p_{k'}$ of P_n^2 whence

$$-\sum_{l=1}^{l=\rho} \alpha_{l,j} = \pm u_{j'} + b, \qquad -\sum_{l=1}^{l=\rho} \alpha_{l,k} u_l = \pm u_{k'} + b.$$

Hence $\sum_{l=1}^{l=\rho} (\alpha_{l,j} - \alpha_{l,k}) u_l = \pm (u_{k'} - u_{j'})$. This relation on the parameters $u_1, \cdots, u_n$ with integral coefficients does not vanish identically for if $u_1 = u_2 = \cdots = u_n = 1$ it takes the value $3(s_j - s_k) \neq 0$. Hence the assumption that K exists requires that P_n^2 be a particular set on E^3 which proves the theorem for a general set on E^3 and *a fortiori* for a general set P_n^2.

We find in § 6 that the number of distinct types of Cremona transformations with 9 F-points is infinite.* Since the number of possible types of congruence is infinite and no types lead to projectively equivalent sets, the number of points Q conjugate to a given point P under $G_{n,2}$ is infinite when $n \geqq 9$. The types of C_m depend upon integer values and the arrangement in Q_n^2 upon a finite number of permutations so that the operations of $G_{n,2}$ form a discontinuous aggregate.

If $n < 9$, $G_{n,2}$ is finite and its order is $n!$ times the number of ways the set

* Kantor, loc. cit., p. 273, Theorem IX.

P_n^2 can be congruent to Q_n^2 under C_m. When $n = 6$, we can use C_1, a collineation, C_2, C_3, C_4, and C_5 in $\binom{6}{0}$, $\binom{6}{3}$, $\binom{6}{4}\binom{2}{1}$, $\binom{6}{2}$, and $\binom{6}{0}$ ways respectively whence the order of $G_{6,2}$ is $6!\,72$. When $n = 7$, we can use C_1, C_2, C_3, C_4, C_5, D_4', D_5, D_6, D_7, D_8 in $\binom{7}{0}$, $\binom{7}{3}$, $\binom{7}{4}\binom{3}{1}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{6}$, $\binom{7}{6}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{1}\binom{6}{4}$, $\binom{7}{3}$, $\binom{7}{0}$ ways respectively or 2.288 ways in all. Since P_7^2 and Q_7^2 congruent under D_8 are projective we find only 288 projectively distinct types of congruence whence $G_{7,2}$ is of order $7!\,288$. Similarly when $n = 8$ we find by using in all possible ways the transformations of the table in § 2 that the number of ways of congruence is 2.8640 which again give rise to only 8640 projectively distinct ways due to the property of E_{17} noted in § 2.

(8) *The extended group $G_{n,2}$ in $\Sigma_{2(n-4)}$ is infinite and discontinuous if $n \geqq 9$. If $n = 6, 7, 8$ the order of $G_{n,2}$ is* $6!\,72$, $7!\,288$, $8!\,8640$ *respectively.*

From the form of the generators of $G_{n!}$ and of A we see that if only the generators of $G_{(n-1)!}$ be retained and if A be applied to only the first $n-1$ points of P_n^2, a subgroup $g_{n-1,2}$ of $G_{n,2}$ is obtained which is isomorphic with the $G_{n-1,2}$ derived from P_{n-1}^2. Hence

(9) *The $G_{n,2}$ contains subgroups isomorphic with $G_{n',2}$ where $n' < n$.*

In general this isomorphism is simple and its existence can be seen from the fact that the subgroup $g_{n',2}$ is precisely the $G_{n',2}$ of a set $P_{n'}$ in the $\Sigma_{2(n'-4)}$ lying in $\Sigma_{2(n-4)}$ and determined by equating to zero the first two coördinates of $p_{n'+1}, \cdots, p_n$. An exceptional case occurs when $n' = 5$. Let us take for convenience $n = 6$. The extended group $G_{5,2}$ in Σ_2 is merely a $G_{5!}$. For even though there are 16 ways under which two sets P_5^2 and Q_5^2 can be congruent by using the C_1, the $\binom{5}{3}$ C_2's, and the $\binom{5}{1}$ C_3's these modes of congruence do not lead to projectively distinct sets, e. g., if P_5^2 and Q_5^2 are congruent under A they are projective in the order (45). But if they be enlarged to sets P_6^2 and Q_6^2 congruent under A by adding a pair of ordinary corresponding points the sets are no longer projective in any order. Thus $G_{6,2}$ contains subgroups $g_{5,2}$ of order $5!\,16$ which are in 16 to 1 isomorphism with $G_{5,2}$ of order $5!$ Other cases of exception occur in dropping from $G_{9,2}$ to $G_{8,2}$ and from $G_{8,2}$ to $G_{7,2}$. Here the isomorphism is 2 to 1.

(10) *The index of the subgroup $G_{n-1,2}$ of $G_{n,2}$ is in general the number of rational curves that are determined by points of P_n^2 increased by the number of points.*

For a particular group $G_{n-1,2}$ is determined by isolating a point of P_n^2. By applying the transpositions of $G_{n!}$ this point is transformed into any point. By applying A and then the transpositions there is obtained the set of lines on two of the points. From these in turn there is obtained in the same way the set of conics on five of the points, etc. An exception occurs when $n = 7$ or $n = 8$ for then the rational curves on P_n^2 are paired under the transformations D_8 or E_{17} which leads to the identity in $G_{n,2}$. Here the index is half the usual number.

(11) *The index of each group of the series* $g_{5,2}$, $G_{6,2}$, $G_{7,2}$, $G_{8,2}$, $G_{9,2}$, $\cdots$, $G_{n,2}$ *considered as a subgroup of the following one is* 27, 28, 120, ∞, $\cdots$, ∞ *respectively.*

In order to identify the finite groups $G_{6,2}$, $G_{7,2}$, $G_{8,2}$ with certain known groups let us begin with $G_{7,2}$. For $i, j, k = 1, \cdots, 7$ let I_{0i} denote that involution of $G_{7,2}$ determined by C_5 with F-points at the points of P_7 other than p_i; let I_{ik} denote that involution of $G_{7,2}$ determined by interchanging p_i and p_k; and let I_{0ijk} denote that involution of $G_{7,2}$ determined by C_2 with F-points at p_i, p_j, p_k. These 63 involutions form a conjugate set of generators of $G_{7,2}$. For they include the generators of (6) and these generators permute the 63 involutions transitively. Indeed under the transpositions the sets I_{0i}; I_{ik}; I_{0ijk} are separately permuted while the element $A = I_{0123}$ leaves I_{12}, I_{45}, I_{01}, I_{0145} unaltered and interchanges the pairs (I_{14}, I_{0234}), (I_{0456}, I_{07}). Let us further denote the pairs of rational curves on P_7^2 as follows: $Q_{0i} \equiv \overline{p_i}^{\,0}\ \overline{p_i^2\, p_j\, p_k \cdots}^{\,3}$; $Q_{ij} \equiv \overline{p_i\, p_j}^{\,1}\ \overline{p_k\, p_l \cdots}^{\,2}$. Then it is easy to verify that for $i, j, k \cdots = 0, 1, \cdots, 7$ the involutions permute the pairs of curves as follows:

$$I_{ij} \;:\quad (Q_{ij})\,(Q_{ik},\, Q_{jk})\,(Q_{kl});$$

$$I_{ijkl}\!:\quad (Q_{ij},\, Q_{kl})\,(Q_{lm})\,(Q_{mn},\, Q_{op})\,.$$

If we use the base notation as set forth in F. G., II,* for the finite geometry associated with the odd and even theta functions in $p = 3$ variables we find in S_{2p-1} a set of 63 points $P_{ij} = P_{klmnop}$, $P_{ijkl} = P_{mnop}$, each of which determines an involution I_{ij}, I_{ijkl} and a set of 28 O quadrics Q_{ij}. These involutions generate the group of the null system in S_{2p-1}. According to F. G., II (10), they permute the O quadrics precisely as the involutions above permute the pairs of curves. Since the orders of the two groups are the same, $G_{7,2}$ is simply isomorphic with the group of the null system or the group of the double tangents of a quartic.

This last connection can be established by mapping the plane E_x of P_7^2 upon a plane E_y by cubics on P_7^2. A point y of E_y corresponds to two points on E_x. These coincide along a sextic curve with double points at P_7^2 which maps into a quartic curve C^4 on E_y. The 28 degenerate cubics map into the 28 double tangents of C^4 in such a way that the seven which correspond to Q_{0i} form an Aronhold system. The operations of $G_{7,2}$ transform P_7^2 into Q_7^2 and transform the net of cubics and its degenerate curves on P_7^2 into the net and degenerate curves on Q_7^2 but of course they leave C^4 unaltered. Thus to

* Two earlier papers of the writer are cited from time to time: *Finite geometry and theta functions*, these Transactions, vol. 14 (1913), p. 241; and *An isomorphism between theta characteristics and the* $(2p+2)$*-point*, Annals of Mathematics (1916). They are referred to as F. G., I and F. G., II respectively.

the 7!288 operations of $G_{7,\,2}$ there correspond the 7! ways in which any one of the 288 Aronhold systems may be arranged.

The subgroup of $G_{7,\,2}$ which leaves Q_{07} unaltered is $G_{6,\,2}$. It is generated by the 36 involutions I_{ik}, I_{07}, I_{0ijk} $(i, j, k = 1, \cdots, 6)$, and being of index 28 under $G_{7,\,2}$ has the order 51840. That it is isomorphic with the group of the 27 lines on a cubic surface $C^3(y)$ is evident either from the fact that cubic curves on P_6^2 map E_x upon $C^3(y)$ with isolated lines or that the quartic tangent cone from a point y (the map of p_7) has for double tangent planes the tangent plane at y and the 27 planes from y to the 27 lines of $C^3(y)$. It is clear from the sample I_{07} that the 36 conjugate generators are those which effect the interchange of opposite lines of the 36 double sixes of $C^3(y)$.

In the case of $G_{8,\,2}$ for subscripts $i, j, k, \cdots = 1, \cdots, 8$ let us denote by I_{ij}, I_{0ijk}, I_{0ij9}, and I_{i9} respectively those involutions of $G_{8,\,2}$ determined by the transposition $(p_i\, p_j)$, the C_2 with F-points at p_i, p_j, p_k, the C_5 with ordinary points at p_i, p_j, and the D_8 with ordinary point at p_i. It can be shown as above that these 120 involutions constitute a conjugate set of generators of $G_{8,\,2}$. Let us denote the 120 pairs of rational curves on P_8^2 as follows:

$$Q_{0i9} \equiv \overline{p_i}^{\,0}\ \overline{p_i^3\, p_j^2\, p_k^2 \cdots}^{\,6}; \qquad Q_{ij9} \equiv \overline{p_i\, p_j}^{\,1}\ \overline{p_i\, p_j\, p_k^2\, p_l^2 \cdots}^{\,5};$$

$$Q_{ijk} \equiv \overline{p_l\, p_m \cdots}^{\,2}\ \overline{p_i^2\, p_j^2\, p_k^2\, p_l\, p_m \cdots}^{\,4}; \qquad Q_{0ij} \equiv \overline{p_i^2\, p_k\, p_l \cdots}^{\,3}\ \overline{p_j^2\, p_k\, p_l \cdots}^{\,3}.$$

On the other hand in the finite geometry associated with the theta functions for $p = 4$ there is, for subscripts $i, j, k, \cdots = 0, 1, \cdots, 9$, a set of 126 E quadrics like $Q_{ijklm} = Q_{nopq}$, and a set of 10 E quadrics like Q_i. Also there are 120 O quadrics like Q_{ijk}, 45 points like P_{ij} and 210 points like P_{ijkl}. If we isolate a definite E quadric like Q_0 we find that the group which leaves this quadric unaltered is generated by a conjugate set of 120 involutions determined by the 120 points outside the quadric. According to F. G., II (8), these points comprise the 36 of type P_{ik} and the 84 of type P_{0ijk} $(i, j, k = 1, \cdots, 9)$. Moreover these involutions permute the O quadrics precisely as the 120 involutions above permute the pairs of curves on P_8^2. Since the orders are the same, the $G_{8,\,2}$ and the group of Q_0 in the finite geometry are simply isomorphic.

This connection with theta functions of genus 4 can also be established by mapping the plane E_x upon a quadric cone $Q^2(y)$ in S_3 by sextic curves with double points at P_8^2.* To a point x there corresponds one point y, to a point y there corresponds two points x which coincide along a 9-ic curve in E_x with triple points at P_8^2. Cubics on P_8^2 map into the generators of Q^2 and the 9-ic curve maps into a sextic of genus 4 on this cone. The 120 degenerate

* In regard to these maps cf. Wiman, *Zur Theorie . . . birationalen Transformationen in der Ebene*, Mathematische Annalen, vol. 48, p. 195.

sextics map into the 120 tritangent planes of this sextic and are therefore associated with the odd theta functions of genus 4. Since P_8^2 has only 8 absolute constants the moduli of the functions are subject to two conditions, one of which is that the functions are Riemannian and the other that the corresponding normal curve is on a nodal quadric. Hence

(12) $G_{6,\,2}$ *is isomorphic with the group of the lines on a cubic surface,* $G_{7,\,2}$ *with the group of the double tangents of a plane quartic curve, and* $G_{8,\,2}$ *with the group of the tritangent planes of a space sextic of genus* 4 *on a quadric cone.*

These three Cremona groups are defined respectively in Σ_4, Σ_6, and Σ_8. Though collineation groups isomorphic with them occur in spaces of lower dimension yet the Cremona groups have a certain initial advantage in that they are in direct algebraic relation with the corresponding geometrical configuration. This advantage appears in § 8 in connection with the invariants of the quartic.

The $G_{n,\,2}$ is a special case of the extended group $G_{n,\,k}$ of the set P_n^k in S_k which is defined in the next paragraph and some of its properties appear hereafter from this point of view.

4. Congruence of sets P_n^k in S_k. The extended group $G_{n,\,k}$ of P_n^k

The theorem that the general Cremona transformation in S_2 can be expressed as a product of quadratic transformations has no analogue for spaces of greater dimension. The possible existence of F-curves, F-surfaces, etc., for the general Cremona transformation C_m^k of order m in S_k so complicates the theory that very few of the relatively simple properties of C_m^2 can be extended to C_m^k. There is however a class of C_m^k's which has properties entirely analogous to C_m^2, namely the class of regular transformations defined below.

The general quadratic transformation in S_2 can be regarded as the product of the quadratic involution A, $x_1' = 1/x_1$, $x_2' = 1/x_2$, $u' = 1/u$, and a projectivity. When projective properties of the transformation are under consideration the projectivity is an unessential factor. So in S_k we shall build up transformations with involutions A of order k of the type $x_i' = 1/x_i$, $u = 1/u$ $(i = 1, \cdots, k)$. Such an involution A is uniquely determined by its $k + 1$ F-points and any corresponding pair in general position and we shall suppose that a given involution is so defined. The product of an involution A and a projectivity will be called an *element* A. In case the F-points of A are drawn from a given set this will be indicated by giving A corresponding subscripts. We shall define a *regular Cremona transformation in* S_k to be one which can be expressed as a product of elements A. If for two such transformations the products can be formed in such a way that the number of elements A in each is the same and the arrangement of the F-points in forming

the product is the same then we shall say that the two are of the same *type*. Since the inverse of a regular transformation is likewise regular and the product of two regular transformations is regular we see that

(13) *The regular Cremona transformations in S_k form a group, the regular Cremona group in S_k.*

We shall understand by an F-point of a regular transformation, as distinguished from a point on an F-space, a point such that the correspondents of points consecutive to it lie on an f-space of dimension $k-1$. Though the regular transformations may have F-spaces of dimension greater than zero, as do the involutions A themselves, yet they have the property in common with C_m^2's that they are determined by their two sets of ρ F-points and by $k+2-\rho$ ordinary pairs if $\rho < k+2$. This statement is true for a projectivity and for an element A. If we assume that it is true for a regular transformation C which can be expressed as a product of $\mu - 1$ elements A then it is easy to see that it is true of the product of C and an element A and therefore holds generally.

That, as assumed above, the number of F-points of the regular transformation C_m^k is the same as that of its inverse $(C_m^k)^{-1}$ can be proved by a similar induction. Let $C_m^k = CA$ where C which sends x into x' has the set $p_1, \cdots, p_\sigma$ of F-points and C^{-1} has the set $q_1, \cdots, q_\sigma$ of F-points. Let A which sends x' into x'' have F-points, $r_1, \cdots, r_{k+1}$ of which τ are found in the set q and the remaining $k+1-\tau$ under C^{-1} correspond to ordinary points of C; and let $s_1, \cdots, s_{k+1}$ be the F-points of A^{-1}. Then the number of F-points of C_m^k is $\rho = (\sigma) + (k+1-\tau) - \mathrm{X}$ where X is the number of F-points of C which correspond to f-spaces of A^{-1} or it is the number of F-points of C which form with F-points of A^{-1} ordinary pairs of the product. The F-points of $(C_m^k)^{-1}$ are those of $A^{-1}C^{-1}$. For this product the numbers σ and $k+1$ interchange while τ and X remain the same whence ρ is the same.

A general set of points P_n^k will be said to be *congruent* to a set Q_n^k under the regular Cremona transformation C_m^k in S_k with $\rho \leqq n$ F-points if the F-points of C_m^k and of $(C_m^k)^{-1}$ are found in P_n^k and Q_n^k respectively and if the remaining $n-\rho$ of the points in each set form $n-\rho$ corresponding pairs of C_m^k. The *type* of congruence is determined by the type of C_m^k. If Q_n^k is projective to $Q_n'^k$ then P_n^k is congruent to $Q_n'^k$ under the transformation $C_m^k\pi$ of the same type as C_m^k. Thus congruence is both a mutual and a projective property of the sets.

If the set P_n^k is congruent to Q_n^k under C_m^k and if Q_n^k is congruent to R_n^k under D_μ^k then P_n^k is congruent to R_n^k under a properly arranged product $C_m^k \cdot D_\mu^k$. The argument used above for the product CA if account be taken of ordinary corresponding pairs under both C and A proves this statement for the case when D_μ^k is an element A. By expressing D_μ^k as a product of elements

A and by using the less general statement as a lemma the original statement can be proved.

If the type of C_m^k be given it can be expressed as a product of elements A. If two sets are congruent under an element A the coördinates of the one can be expressed rationally in terms of the coördinates of the other. Moreover this expression is unique if $k+1$ points of each set be placed at a given base. Evidently the same property holds for a product of elements A so that if P_n^k is congruent in some order to Q_n^k the coördinates of Q_n^k are rational in those of P_n^k.

If we map the sets P_n^k upon the points P of a space $\Sigma_{k(n-k-2)}$ as suggested in I, § 6 (53), then most of the conclusions above are embraced in the theorem:

(14) *If the sets P_n^k and Q_n^k are congruent in some order under regular transformation in S_k their maps P and Q in $\Sigma_{k(n-k-2)}$ are corresponding points under a Cremona transformation in Σ. Two sets congruent to a third are congruent to each other. The aggregate of projectively distinct sets Q_n^k congruent in some order to a given set P_n^k is mapped in Σ by an aggregate of points Q which form a conjugate set under the extended Cremona group $G_{n,\,k}$ in Σ.*

By using considerations entirely similar to those of § 3 the theorems (6) and (9) of § 3 can be generalized as follows:

(15) *The extended group $G_{n,\,k}$ in $\Sigma_{k(n-k-2)}$ can be generated by the $G_{n!}$ of P_n^k (whose generators are given in I, § 7 (64)) and the involutory transformation $\bar{A}$ obtained by inverting the variables in Σ. The $G_{n,\,k}$ contains subgroups isomorphic with $G_{n',\,k}$ if $n' < n$.*

This isomorphism is simple except in the cases where congruence of sets $P_{n'}^k$ implies projectivity in some order. In case $n' = k+3$ the isomorphism is 1 to 2^{k+2}.

The generator $\bar{A}$ determined by the element A is conjugate under $G_{n!}$ to the involution of type $\bar{A}$ determined by any $k+1$ of the points of P_n^k and it may therefore be replaced by any such involution. From the form of the associated sets P_n^k, Q_n^{n-k-2} given in I, § 6, p. 182, it is clear that the groups $G_{n!}$ coincide for the two sets and the generator $\bar{A}_{1,\,2,\,\ldots,\,k+1}$ of the one coincides with the generator $\bar{A}_{k+2,\,\ldots,\,n}$ of the other. Hence

(16) *The extended groups $G_{n,\,k}$ and $G_{n,\,n-k-2}$ of the associated point sets P_n^k and Q_n^{n-k-2} coincide.*

The various point sets can be arranged so that those whose norm

$$N = 2k + 2 - n$$

is zero lie in the principal diagonal of the array and those whose norms differ in sign are symmetrical with regard to this diagonal. An examination of this array with (15) and (16) in mind leads to the result:

(17) *The group $G_{n,\,k}$ contains subgroups isomorphic with $G_{n',\,k'}$, if either $k' \leqq k$ and $n' - k' < n - k$ or $k' < n - k - 2$ and $n' - k' \leqq n - k$.*

We prove in § 6 that the number of types of congruence for P_9^2 and P_8^3 is infinite. Hence the number of types of congruence for sets of the array

$$\begin{array}{cccc} P_6^2 & P_7^2 & P_8^2 & \cdots \\ P_7^3 & P_8^3 & \cdots & \cdots \\ P_8^4 & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

beyond P_8^2 in the first row and beyond P_7^3 in the second row is infinite. According to (16) this is true also of sets in the first column below P_8^4 and sets in the second column below P_7^2. Then according to (15) it is true of all the sets in all the rows beyond P_8^2, P_7^3, and P_8^4.

The proof in § 3 that the sets P_n^2 $(n \geqq 9)$ congruent under C_m^2 are projectively distinct was based first on the fact that if P_n^2 was on an elliptic norm-curve the congruent set was on a birationally equivalent elliptic norm-curve, and second on the fact that C_m^2 had F-points of different types. Since the element A transforms elliptic $E^{(k+1)}$'s on P_n^k into elliptic $E^{(k+1)}$'s on Q_n^k, the transformation C_m^k with F-points at P_n^k has the same property. Moreover we shall see in § 5 (12) that no C_m^k has symmetrical F-points except those determined by the following sets for $n > k+3$: P_6^2; P_7^2, P_7^3; P_8^2, P_8^4. Thus in all further cases congruence of general sets implies non-projectivity.

(18) *The only finite groups* $G_{n,\,k}$ $(n > k+3)$ *are the* $G_{6,\,2}$, *the* $G_{7,\,2} = G_{7,\,3}$, *and the* $G_{8,\,2} = G_{8,\,4}$ *which are identified in* § 3 (9). *All other groups* $G_{n,\,k}$ *are infinite and discontinuous.*

In the next two paragraphs groups isomorphic with $G_{n,\,k}$ are studied and the results are used to develop further properties of the regular transformations C_m^k and of the groups $G_{n,\,k}$.

5. The collineation group $g_{n,\,k}$

There is determined by the regular transformations associated with the set P_n^k a collineation group $g_{n,\,k}$ isomorphic with the Cremona group $G_{n,\,k}$ in $\Sigma_{n(n-k-2)}$. An element of $g_{n,\,k}$ describes the effect on spreads in S_k of a regular transformation C whose F-points are found in P_n^k. If a spread of order x_0 in S_k has an x_i-fold point at p_i it is transformed by C into a spread of order x_0' which has an x_j'-fold point at q_j $(i, j = 1, \cdots, n)$. Given the type of C and the order of the set P_n^k with reference to Q_n^k this relation between x and x' must be unique, linear, and reversible, whence it is a proper collineation. Since C can be generated by a sequence of elements A, the $g_{n,\,k}$ is generated by

$$g_{n!}: \quad x_0' = x_0, \qquad x_i' = x_j; \qquad\qquad (i, j = 1, \cdots, n);$$

$$\begin{aligned} A: \quad x_0' &= kx_0 - x_1 - x_2 - \cdots - x_{k+1}, \\ x_1' &= (k-1)x_0 - x_2 - \cdots - x_{k+1}, \end{aligned}$$

$$
\begin{aligned}
x_2' &= (k-1)x_0 - x_1 \qquad - \cdots - x_{k+1}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{k+1}' &= (k-1)x_0 - x_1 - x_2 - \cdots \qquad ,\\
x_{k+2}' &= \qquad\qquad\qquad\qquad - (-1)\,x_{k+2}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n' &= \qquad\qquad\qquad\qquad\qquad - (-1)\,x_n\,.
\end{aligned}
$$

The general transformation can be written in a form similar to that used for ternary Cremona transformations

$$
(20)\qquad
\begin{aligned}
x_0' &= m x_0 - \textstyle\sum_{i=1}^{i=n} r_i\, x_i\,,\\
x_j' &= s_j\, x_0 - \textstyle\sum_{i=1}^{i=n} \alpha_{ji}\, x_i
\end{aligned}
\qquad (j = 1, \cdots, n).
$$

The coefficients of the element (20) indicate that C transforms an S_{k-1} into an m-ic spread with an s_j-fold point at q_j and that p_i is an F-point whose corresponding f-spread of order r_i has an α_{ji}-fold point at q_j $(i, j = 1, \cdots, n)$. Here we make the convention that for a point p_i, $x_i = -1$, $m = x_j = 0$ $(j \neq i)$. For an ordinary point p_i which corresponds to q_j, $\alpha_{ji} = -1$, $\alpha_{ki} = 0$ $(k \neq j)$, $\alpha_{jk} = 0$ $(k \neq i)$, $r_i = s_j = 0$. Each type of transformation C leads to a type of element g and these various types combined with $g_{n!}$ give rise to all the elements of $g_{n,\,k}$. For example in the case of P_7^3 we find by combining cubic transformations A 9 types of elements (20) whose matrices are

C_3

		4	3
	3	− 1	0
4	2	0, − 1	0
3	0	0	1, 0

C_5

		2	4	1
	5	− 1	− 1	0
2	4	− 2, − 1	− 1	0
4	2	− 1	− 1, 0	0
1	0	0	0	1

C_7

		1	3	3
	7	− 3	− 2	− 1
1	6	− 2	− 2	− 1
3	4	− 2	− 1	0, − 1
3	2	− 1	0, − 1	0

C_7'

		6	1
	7	− 2	0
6	4	− 2, − 1	0
1	0	0	1

C_{13}

		3	4
	13	-4	-3
3	8	$-3, -2$	-2
4	6	-2	$-2, -1$

C_{11}

		1	4	2
	11	-4	-3	-2
1	8	-3	-2	-2
4	6	-2	$-1, -2$	-1
2	4	-2	-1	$-1, 0$

C_9

		3	3	1
	9	-3	-2	-1
3	6	-2	$-2, -1$	-1
3	4	$-2, -1$	-1	0
1	2	-1	0	0

C_9'

		1	6
	9	-4	-2
1	8	-3	-2
6	4	-2	$0, -1$

C_{15}:

		7
	15	-4
7	8	$-3, -2$

.

Here the numbers outside of the diagram indicate the number of rows and columns of the matrix. The corresponding rectangles of the matrix are to be filled with the numbers given within. If two numbers are given the principal diagonal of the corresponding square is to be filled with the first of the two and the other places with the last of the two numbers.

The type C_{15} is a symmetric type listed in (30). Congruence under this type implies projectivity. It pairs off the remaining types by multiplication the pairs being easily located above. It is itself paired with the collineation type of C. The number of elements of the various types obtained by permuting the rows (or columns) is $2\binom{7}{4}$, $2\binom{7}{1}\binom{6}{2}$, $2\binom{7}{1}\binom{6}{3}$, $2\binom{7}{1}$, $2\binom{7}{0}$, i. e., 576 in all whence the order of $g_{7,3}$ is 2.288.7!. Allowing for the projectivity under C_{15} the order of $G_{7,3} = G_{7,2}$ is only 288.7!. Thus $g_{7,3}$ and $G_{7,3}$ are in 2 to 1 isomorphism.

Any invariant form of $g_{n,k}$ is symmetric in $x_1, \cdots, x_n$. It is easy to verify that the linear form

$$L \equiv (k+1)x_0 - (x_1 + \cdots + x_n)$$

is unaltered by A and therefore is an absolute invariant of $g_{n,k}$. An invariant quadratic form can be combined with L^2 so that no term in x_0^2 appears.

Assuming that it is $\alpha x_0 \sum x_1 + \beta \sum x_1^2 + \gamma \sum x_1 x_2$ and applying A we find that $\alpha = -2(k^2 - 1)$, $\beta = k(k+3)$, $\gamma = 2(k-1)$. By adding a proper multiple of L^2 the terms in $x_0 \sum x_1$ and $\sum x_1 x_2$ are eliminated and the invariant quadratic form is

$$M \equiv (k-1)x_0 - (x_1^2 + \cdots + x_n^2).$$

The $g_{n,k}$ has an invariant point also. For an S_{k-1} becomes under the involution A in S_k a $k$ic spread with $(k-1)$-fold points so that a $(k+1)$ic spread with $(k-1)$-fold points at P_n^k is transformed into a spread of the same sort. Hence the point

$$O \equiv k+1, \quad k-1, \quad \cdots, \quad k-1,$$

is invariant under $g_{n,k}$.

(21) *The $g_{n,k}$ has for absolute invariant the quadratic form M, as well as the point O and linear form L which are pole and polar as to M.*

There are three particular cases of interest here, namely those for which O and L are incident. This occurs when $(k+1)^2 - n(k-1) = 0$ or $n = k + 3 + 4/(k-1)$. Since n must be an integer, $k - 1 = 1, 2, 4$; $n = 9, 8, 9$. These cases P_9^2, P_8^3, P_9^5 have the further peculiarity that they are the only point sets which lie on a unique elliptic norm curve. For the $E^{(k+1)}$ in S_k has $(k+1)^2$ constants and it is $(k-1)$ conditions that it be on a point whence n must be $(k+1)^2/(k-1)$ if the number of $E^{(k+1)}$'s on P_n^k is to be finite.

The generator A and the transpositions of $g_{n,k}$ are such that for them $m - 1$ and the s_j contain $k - 1$ as a factor. If this is true of two elements of $g_{n,k}$ with coefficients m, m'; s_j, s_j'; r_i, r_i'; $\alpha_{ji}, \alpha_{ji}'$ it is true of the product. For $m'' = mn' - \sum r_i' s_i$ and

$$m'' - 1 = (m-1)(m'-1) + (m-1) + (m'-1) - \sum r_i' s_i;$$

and $s_j'' = s_j' m - \sum_{i=1}^{i=n} \alpha_{ji}' s_i$. Hence this factor will always appear in the m and s_j of (20) and it will be convenient to set

$$m - 1 = (k-1)\mu, \qquad s_j = (k-1)\sigma_j, \qquad r_i = \rho_i.$$

Then the general element of $g_{n,k}$ has the matrix

$$(22) \qquad \begin{pmatrix} (k-1)\mu + 1 & -\rho_1 & \cdots & -\rho_n \\ (k-1)\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ (k-1)\sigma_n & -\alpha_{1n} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

This matrix takes the following more symmetric form under the substitution $\bar{x}_0 = i\sqrt{k-1}\, x_0$:

$$(23)\qquad \begin{pmatrix} (k-1)\mu+1 & -i\sqrt{k-1}\,\rho_1 & \cdots & -i\sqrt{k-1}\,\rho_n \\ -i\sqrt{k-1}\,\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ -i\sqrt{k-1}\,\sigma_n & -\alpha_{n1} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

The invariant quadratic form M now is

$$\bar{M} = \bar{x}_0^2 + x_1^2 + \cdots + x_n^2,$$

whence the matrix (23) is orthogonal. Hence the determinant of (23) is $\epsilon = \pm 1$ and the minor of any element is ϵ times the element. The inverse matrix is obtained by interchanging rows and columns. On transforming back to x_0 we find that the matrix inverse to (22) is obtained by interchanging the ρ's with the σ's and the α_{ji} with the α_{ij}. But this inverse matrix itself belongs to an element of $g_{n,\,k}$ which corresponds to the inverse Cremona transformation in S_k. This shows that

(24) *For a regular Cremona transformation in S_k the direct and inverse transformation have the same order and the same number of F-points, and the numbers ρ_i, α_{ji}, σ_j for the one are the numbers σ_j, α_{ij}, ρ_i for the other.*

If we make use of the fact that L, M, and O are absolutely invariant under the element of $g_{n,\,k}$ whose matrix is (22) we get certain relations on the integer coefficients. If in these relations we transpose the coefficients according to (24) new relations are obtained. The entire set of relations is as follows:

$$(25)\qquad \begin{array}{llll} \sum_j \sigma_j &= (k+1)\mu, & \sum_j \sigma_j^2 &= (m+1)\mu, \\ \sum_j \alpha_{ji} &= (k+1)\rho_i - 1, & \sum_j \alpha_{ji}^2 &= (k-1)\rho_i^2 + 1, \\ \sum_j \alpha_{ji}\sigma_j &= \rho_i m, & \sum_j \alpha_{ji}\alpha_{jk} &= (k-1)\rho_i\rho_k; \\ \sum_i \rho_i &= (k+1)\mu, & \sum_i \rho_i^2 &= (m+1)\mu; \\ \sum_i \alpha_{ji} &= (k+1)\sigma_j - 1, & \sum_i \alpha_{ji}^2 &= (k-1)\sigma_j^2 + 1, \\ \sum_i \alpha_{ji}\rho_i &= \sigma_j m, & \sum_i \alpha_{ji}\alpha_{ki} &= (k-1)\sigma_j\sigma_k. \end{array}$$

These relations reduce in S_2 to the well-known set given in Clebsch-Lindemann (loc. cit.). The further facts there proven with regard to the sets of equal numbers ρ_i and σ_j and to the rectangles and squares in the matrix $\|\alpha_{ji}\|$ determined by these sets could be extended readily to the regular transformations in S_k.

It was noted in § 4 that $G_{n,\,k}$ and $G_{n,\,n-k-2}$ were identical since for proper arrangement of the points of P_n^k and Q_n^{n-k-2} the transpositions (ij) of $G_{n!}$ were the same in Σ and the generators $A_{1,\,\ldots,\,k+1}$ and $A_{k+2,\,\ldots,\,n}$ respectively also were the same in Σ. Since $g_{n,\,k}$ and $g_{n,\,n-k-2}$ are isomorphic with these groups they are in natural isomorphism with each other. We

might expect therefore to find that the one is merely a transform of the other. In order to prove this consider the involutions which generate $g_{n,\,k}$. They all are of the point $-\,S_{n-1}$ type. The invariant point of a transposition (ij) is $x_i = -x_j = 1$, $x_l = 0$; the S_{n-1} is $x_i - x_j = 0$. The invariant point of A is $x_0 = x_1 = \cdots = x_{k+1} = 1$, $x_{k+2} = \cdots = x_n = 0$; the S_{n-1} is

$$(k-1)x_0 - x_1 - \cdots - x_{k+1} = 0.$$

Hence the point of each is on L and the S_{n-1} of each is on O or

(26) *Every element of $g_{n,\,k}$ is interchangeable with every element of the one-parameter group of homologies determined by O, L.* Thus if $g_{n,\,k}$ and $g_{n,\,n-k-2}$ are conjugate they are conjugate in infinitely many ways. A collineation T which sends the transpositions of $g_{n,\,k}$ into the corresponding ones of $g_{n,\,n-k-2}$ must have the form

$$T:\quad x_0 = \alpha\bar{x}_0 + \beta\bar{\sigma}, \qquad \bar{\sigma} = \textstyle\sum_i \bar{x}_i$$

$$x_i = \gamma\bar{x}_0 + \epsilon\bar{x}_i + \delta\bar{\sigma}, \qquad i = (1, \cdots, n).$$

If T sends $(k-1)x_0 - x_1 \cdots - x_{k+n}$ into $(n-k-3)\bar{x}_0 - \bar{x}_{k+2} - \cdots - \bar{x}_n$ then $(k-1)\alpha - (k+1)\gamma + \epsilon(n-k-3) = 0$ and

$$\epsilon = (k-1)\beta - (k+1)\delta.$$

If it sends the fixed point of $A_{1,\,\ldots,\,k+1}$ into that of $A_{k+2,\,\ldots,\,n}$ then

$$\alpha + (n-k-1)\beta + \epsilon = 0, \qquad \gamma + (n-k-1)\delta + \epsilon = 0.$$

In order to get a definite form for T let $\delta = 0$ and $\epsilon = k-1$. Then $\gamma = -(k-1)$, $\beta = 1$, and $\alpha = -(n-2)$. These coefficients are valid for all values of n and k with which we are concerned though the theorem (26) admits the three exceptions mentioned under (21). To within a factor of proportionality we get

$$T^{-1}:\qquad \bar{x}_0 = (k-1)x_0 - \sigma,$$

$$\bar{x}_i - \bar{x}_0 = 2x_i.$$

The determinant of T is $2(k-1)^n$ and of T^{-1} is $2^n(k-1)$. If then we form the product T (22) T^{-1}, where (22) is the matrix above, and divide each coefficient by $2(k-1)$ the resulting element is that one of $g_{n,\,n-k-2}$ which corresponds in the above isomorphism to the element (22) of $g_{n,\,k}$.

(27) *The group $g_{n,\,k}$ is transformed into its associated $g_{n,\,n-k-2}$ by the substitution T^{-1}. The element of $g_{n,\,n-k-2}$ which corresponds to the element* (22) *of $g_{n,\,k}$ has the matrix*

$$\begin{pmatrix} (n-k-3)\mu + 1 & -(\mu - \rho_i) \\ (n-k-3)(\mu - \sigma_j) & -(\mu - \rho_i - \sigma_j + \alpha_{ji}) \end{pmatrix}.$$

These corresponding elements of $g_{n,\,k}$ and $g_{n,\,n-k-2}$ lead to the same element in $G_{n,\,k} = G_{n,\,n-k-2}$.

Since the α_{ij} are usually positive integers or zero but may for ordinary points have the value -1 we see that

(28) *For every regular Cremona transformation in S_k the inequalities*

$$\sigma_j + \rho_i \leqq \mu + 1 + \alpha_{ji}$$

are valid. In particular for S_2 $\sigma_j + \rho_i \leqq m + \alpha_{ji}$.

This inequality for S_2 is known but the above proof based on point sets in higher dimensions seems rather interesting.

We shall now determine the types of regular Cremona transformations in S_k with a single symmetrical set of F-points. For the corresponding element of $g_{n,\,k}$, $\rho_i = \sigma_j = \rho$, $\alpha_{ii} = \alpha$, and $\alpha_{ij} = \beta \neq \alpha$. The relations (25) now become

$$n\rho = (k+1)\mu, \qquad n\rho^2 = \mu\,[(k-1)\mu + 2],$$

$$(n-1)\beta + \alpha = (k+1)\rho - 1, \qquad (n-1)\beta^2 + \alpha^2 = (k-1)\rho^2 + 1,$$

$$(n-1)\beta + \alpha = (k-1)\mu + 1, \qquad (n-2)\beta^2 + 2\alpha\beta = (k-1)\rho^2.$$

From the last two $\alpha = \beta \pm 1$ and the set can be replaced by

$$n\beta^2 \pm 2\beta = (k-1)\rho^2, \qquad (k \pm 1)\rho = (k-1)\mu + 2,$$

$$n\beta \pm 1 = (k-1)\mu + 1, \qquad n\rho = (k+1)\mu$$

Eliminating ρ from the second and fourth of these we have

$$[(k+1)^2 - n(k-1)]\,\mu = 2n.$$

Since n, k, μ are positive integers we have

$$1 \leqq [(k+1)^2 - (k-1)n] \leqq 2n$$

whence $k + 1 \leqq n \leqq k + 3 + 3/(k-1)$, or $k + 1 \leqq n < k + 6$. Taking for n values from $k + 6$ to $k + 1$ in order we find that if

(a) $n = k + 6$, $\mu = (2k + 12)/(-3k + 7)$ whence since $\mu > 0$ and $k \leqq 2$, $k = 2$, $\mu = 16$, $n = 8$, $\rho = 6$, $\beta = 2$, $\alpha = 3 : P_8^2$;

(b) $n = k + 5$, $\mu = (2k + 10)/(-2k + 6)$ whence $k = 2$, $\mu = 7$, $n = 7$, $\rho = 3$, $\beta = 1$, $\alpha = 2 : P_7^2$;

(c) $n = k + 4$, $\mu = (2k + 8)/(-k + 5)$ whence

(c_1) $k = 4$, $\mu = 16$, $n = 8$, $\rho = 10$, $\beta = 6$, $\alpha = 7 : P_8^4$;

(c_2) $k = 3$, $\mu = 7$, $n = 7$, $\rho = 4$, $\beta = 2$, $\alpha = 3 : P_7^3$;

(c_3) $k = 2, \mu = 4, n = 6, \rho = 2, \beta = 1, \alpha = 0 : P_6^2$;
(d) $n = k + 3,\ \mu = (k + 3)/2,\ \rho = (k + 1)/2,\ \beta = (k - 1)/2,$
$\alpha = (k + 1)/2 : P_{k+3}^k$ k odd;
(e) $n = k + 2,\ \mu = 2(k + 2)/(k + 3) = 2 - 2/(k + 3)$ which is impossible;
(f) $n = k + 1, \mu = 1, \rho = 1, \beta = 1, \alpha = 0 : P_{k+1}^k$.

So far as the existence of the corresponding Cremona transformations is concerned we see that the type (f) is the element A, and the types (a), (b), (c_3) were discussed in § 2. The type (d) for $k = 3$ is the C_7' listed at the beginning of this paragraph. All these are well known but the types (c_1),* (c_2), and (d) for general k seem new to the literature. The existence of the types (c_1) and (c_2) is a consequence of the existence of their associated types (a) and (b). To show that (d) exists for every odd value of k let $k=2\nu+1$, let P_{k+3}^k be $p_1, \cdots, p_{2\nu+4}$, and let I_{ij} be the involution A which interchanges p_i, p_j and which has the other $2\nu + 2$ points as F-points. Then it is not hard to verify that the transformation $I_{12} I_{34} \cdots I_{2\nu+3,\, 2\nu+4}$ is of the required type (d). If the properties of the involutory type (d) as known for $k = 3$ be generalized we get a new generalization of the Kummer and Weddle surfaces.

(29) *The only point sets which serve in a symmetrical way as F-points to define a regular Cremona transformation are* P_{k+1}^k $(k \geqq 2)$; P_{k+3}^k $(k$ *odd*$)$; P_6^2; P_7^2, P_7^3; *and* P_8^2, P_8^4.

The elements of $g_{n,\,k}$ corresponding to these symmetric types are necessarily involutory. In most of the cases this involution is the harmonic perspectivity I determined by O and L whose matrix is

$$\mathrm{I}:\ \left(\begin{array}{ccc} n(k-1)+(k+1)^2 & -2(k+1) & -2(k+1) \\ 2(k-1)^2 & -[(k+1)^2-(n-2)(k-1)] & -2(k-1) \\ 2(k-1)^2 & -2(k-1) & -[(k+1)^2-(n-2)(k-1)] \end{array}\right).$$

If this is to leave L absolutely unaltered all the coefficients must be divided by $[(k + 1)^2 - n(k - 1)]$. For P_8^2, P_8^4 and P_7^2, P_7^3 this factor is 1 and I belongs to $g_{n,\,k}$. For P_{k+3}^k the factor is 4 and again I belongs to $g_{n,\,k}$. For P_{k+1}^k and P_6^2 however the symmetrical element is found in the conjugate set of generators and I does not belong to $g_{n,\,k}$.

(30) *The element of* $g_{n,\,k}$ *corresponding to the above symmetrical types is the involution I determined by O and L when* $\alpha > \beta$; *when* $\alpha < \beta$ *it is found in the conjugate set of generators.*

The transpositions of $g_{n!}$ and the element A in (19) all have the determinant -1 whence

* For the type (c_1) cf. Conner, loc. cit.

(31) *The group $g_{n,k}$ has an invariant subgroup of index two which consists of all of its elements of determinant unity.*

6. THE GROUP $e_{n,k}$

The group $g_{n,k}$ of § 5 represented the effect of regular Cremona transformations in S_k upon spreads of dimension $k-1$. The group $e_{n,k}$ of this paragraph represents the transition from a set P_n^k given by its elliptic parameters on an elliptic norm curve E^{k+1} in S_k to a congruent set Q_n^k similarly given on the same curve. This requires in general that P_n^k be a special set but the resulting transformations on the parameters are linear and their group $e_{n,k}$ is isomorphic with $G_{n,k}$. We shall find that in certain cases this new group is very useful.

The element A in S_k transforms a curve of order g with h_i-fold points at p_i $(i=1,\cdots,k+1)$ into a curve of order $(k+1)g-k\sum_{i=k}^{i=k+1}h_i$ with $[g+h_i-\sum_{j=1}^{j=k+1}h_j]$-fold points at q_i. It therefore transforms an elliptic curve E^{k+1} with simple points at P_n^k into an elliptic curve E'^{k+1} with simple points at Q_n^k. Since E and E' have the same modulus and are normal curves in S_k each invariant under a collineation group of order $2(k+1)^2$ there will be a collineation K—any one of $2(k+1)^2$ possible collineations—which will transform E' back into E. Let v be the elliptic parameter on E such that $v_1+\cdots+v_{k+1}\equiv 0 \pmod{\omega}$ is the condition that $k+1$ points of E lie on an S_{k-1}. Let $v=u_1,\cdots,u_n$ be the parameters on E of P_n^k whose first $k+1$ points are the F-points of A. Let the point v of E be transformed by A into the point v of E' and let the point v of E' be transformed by K into the point v' of E. Then the $k+1$ points $v_1',\cdots,v_{k+1}'$ lie on an S_{k-1} if the $k+1$ points $v_1,\cdots,v_{k+1}$ lie on a $k$ic spread with $(k-1)$-fold points at $u_1,\cdots,u_{k+1}$. Hence $v_1'+\cdots+v_{k+1}'\equiv v_1+\cdots+v_{k+1}+(k-1)(u_1+\cdots+u_{k+1})$. Thus $v'=v+(u_1+\cdots+u_{k+1})(k-1)/(k+1)$ represents the effect of AK upon an ordinary point of A. But for an F-point of A, say $v=u_i$ on E, the corresponding point on E' is not $v=u_i$ but rather the corresponding point q_i of Q_n^k whose parameter is $v_i=u_i-(u_1+\cdots+u_{k+1})$. For this point therefore $v_i'=u_i-(u_1+\cdots+u_{k+1})2/(k+1)$, $(i=1,\cdots,k+1)$. Since Q_n^k is congruent to P_n^k under A the transform of Q_n^k by the collineation K is also congruent to P_n^k whence the set on E^{k+1} determined by parameters u', where

$$\text{(32)}\quad \begin{aligned} A:\quad u_i' &\equiv u_i-(u_1+\cdots+u_{k+1})2/(k+1) && (i=1,\cdots,k+1),\\ u_j' &\equiv u_j+(u_1+\cdots+u_{k+1})(k-1)/(k+1) && (j=k+2,\cdots,n), \end{aligned}$$

is congruent to the set u on E^{k+1} under the regular transformation of type A.

It should be noted that a set u_i on E is projectively equivalent to the

$2(k+1)^2$ sets, $\pm u_i + \omega/(k+1)$, where ω is any period. If the sign of all the u's be changed, the signs of the u''s in (32) are changed. If the u's be increased by $\omega/(k+1)$, $(u_1 + \cdots + u_{k+1})$ is unaltered and the u''s each are increased by $\omega/(k+1)$. In any case $(u_1 + \cdots + u_{k+1})/(k+1)$ is indeterminate to within $\omega/(k+1)$. Thus (32) might be regarded as a relation between the two sets of $2(k+1)^2$ projectively equivalent P_n^k's, $\pm u_i + \omega/(k+1)$ and $\pm u_i' + \omega/(k+1)$. But either set can be determined by a sample and it is simpler to drop the congruence sign and to use the equality sign. Then we have merely a linear transformation from the set u to the congruent set u'.

The process by which we combine elements A to obtain sets congruent to P_n^k is represented on E^{k+1} by the process of combining linear transformations of type A and a change of the order of the points is represented by a change of the order of the parameters whence

(33) *The totality of sets Q_n^k congruent in some order to a given set P_n^k on E^{k+1} with paramaters $u_1, \cdots, u_n$ is obtained on E^{k+1} by effecting on the parameters u the operations of the group $e_{n,\,k}$ generated by transpositions of the parameters and by A in (32). These generators form part of a conjugate set of involutory elements. The group $e_{n,\,k}$ of linear transformations of determinant ± 1 is isomorphic with $G_{n,\,k}$ and is simply isomorphic with $g_{n,\,k}$.*

This simple isomorphism with $g_{n,\,k}$ is due to the fact that each describes the result of regular transformations C_m^k in S_k.

The transpositions are point-S_{n-1} involutions. The generator A is of the same sort. Every point of $u_1 + \cdots + u_k = 0$ is fixed under A and the point $u_i = 1, u_j = -(k-1)/2$ $(i = 1, \cdots, k+1; j = k+2, \cdots, n)$ is changed in sign under A. The group $e_{n,\,n-k-2}$ associated with Q_n^{n-k-2} is as before in natural isomorphism with $e_{n,\,k}$ and we should expect that the one is the transform of the other in such a way that $e_{n!}$ is the same for both and $A_{1,\,\ldots,\,k+1}$ of $e_{n,k}$ is transformed into $A_{k+2,\,\ldots,\,n}$ of $e_{n,\,n-k-2}$. The transformation

$$T: \qquad u_i = -\bar{u}_i + \bar{\sigma}/(k+1) \qquad (\bar{\sigma} = \bar{u}_1 + \cdots + \bar{u}_n)$$

will send $u_1 + \cdots + u_{k+1}$ into $\bar{u}_{k+2} + \cdots + \bar{u}_n$; and

$$T^{-1}: \qquad \bar{u}_i = -u_i + \sigma/(n-k-1)$$

sends the fixed point of $A_{1,\,\ldots,\,k+1}$ into that of $A_{k+2,\,\ldots,\,n}$. Hence

(34) *The group $e_{n,\,k}$ of P_n^k is transformed by the substitution, $u_i = \bar{u}_i + \bar{\sigma}/(k+1)$ into the group $e_{n,\,n-k-2}$ of Q_n^{n-k-2}. Corresponding elements of the two groups belong to the same element of $G_{n,\,k} = G_{n,\,n-k-2}$.*

An invariant form of $e_{n,\,k}$ must be symmetric in $u_1, \cdots, u_n$. If it admits also the element A it is unaltered by the entire group. If then we test the linear and quadratic forms we find that

(35) *The group* $e_{n,k}$ *has the absolute quadratic invariant*

$$[(k+1)^2 - n(k-1)](u_1^2 + \cdots + u_n^2) + (k-1)(u_1 + \cdots + u_n)^2.$$

For the cases P_9^2, P_9^5; P_8^3 *only has* $e_{n,k}$ *the invariant linear form* $\sigma = u_1 + \cdots + u_n$.

Of the two elements of $g_{n,k}$ and $e_{n,k}$ which are in natural correspondence the former is more explicit concerning the properties of C_m^k while usually, as we shall see, the latter has a more convenient form. It is then worth while to find one element in terms of the other. Let the element of $e_{n,k}$ determined by the element (20) of $g_{n,k}$ be

$$u_j' = \sum_{i=1}^{i=n} \beta_{ji} u_i \qquad (j = 1, \cdots, n). \tag{36}$$

This is determined first from the fact that $k+1$ points of E'^{k+1} on an S_{k-1} correspond to the meets of E^{k+1} with an m-ic spread. This leads to the relation

$$v_1' + \cdots + v_{k+1}' = v_1 + \cdots + v_{k+1} + (k-1)(\rho_1 u_1 + \cdots + \rho_n u_n)$$

or $v' = v + (\rho_1 u_1 + \cdots + \rho_n u_n)(k-1)/(k+1)$. But the parameter v_j on E^{k+1} which on E'^{k+1} determines the point q_j which corresponds to p_j is furnished by the extra meet with E^{k+1} of the f-spread determined by q_j, i. e., $v_j = -\sum_{i=1}^{i=n} \alpha_{ji} u_i$. Substituting this value for v above we find that

$$\beta_{ji} = (k-1)\rho_i/(k+1) - \alpha_{ji}. \tag{37}$$

This determines the element of $e_{n,k}$ in terms of the element of $g_{n,k}$. If on the other hand the β_{ji} are given we can first locate the zero values of ρ_i by noting the columns of β_{ji} whose elements all are zero except one which is unity. From the relations (25) § 5 we find that

$$\sum_j \beta_{ji} = \rho_i [n(k-1) - (k+1)^2]/(k+1) + 1.$$

Thus ρ_i is determined except when P_n^k is P_9^2, P_9^5, or P_8^3. In these cases we can use $\sum_j \beta_{ji}^2 = 2(k-1)(\rho_i^2 - \rho_i) + (3k-1)/(k+1)$. This equation in ρ_i is satisfied by a single positive integer ρ_i greater than zero. Knowing the values ρ_i we find from (37) the α_{ji} and again from (25) § 5 the m and σ_j.

The remainder of this paragraph is devoted to the particular cases of P_9^2, P_9^5, and P_8^3. Since P_9^2 and P_9^5 are associated sets we need to consider only the one. The sets P_9^2 and P_8^3 have an added interest in that they are the first sets for which $G_{n,k}$ is infinite. For these sets also $e_{n,k}$ has the absolute linear invariant σ.

Let us first investigate the conjugate set of irrational absolute invariants typified by $u_i - u_j$ whose vanishing implies that the two points p_i and p_j of P_9^2 on E^3 have become coincident. Of this kind there are 36 conjugate under $e_{9!}$. If on these we effect the transformation A we get a new kind

$u_i + u_j + u_k$ which vanishes if the three points are on a line. There are $\binom{9}{3} = 84$ of these conjugate under $e_{9!}$. A further application of A leads to the type $u_1 + \cdots + u_6$ which vanishes if the six points are on a conic. If A be used again we get the type $2u_1 + u_2 + \cdots + u_8$ which vanishes if a rational cubic with node at u_1 passes through $u_2, \cdots, u_8$; etc., *ad infinitum*. We observe however that if $\sigma = 0$ the third type $u_1 + \cdots + u_6$ and the second type $u_7 + u_8 + u_9$ coalesce corresponding to the geometric fact that if P_9^2 is the base of a pencil of cubics and if three points are on a line then the remaining six points are on a conic. Similarly the fourth and the first type coalesce if $\sigma = 0$, a fact also geometrically evident. Let us say that the coalescent types are *congruent mod.* σ. We have therefore only 120 types incongruent mod. σ and these types are permuted by the elements of $e_{9,2}$. To identify this permutation group let us take the basis notation for the theta functions, $p = 4$. Let the E quadric Q_0 be isolated, let the type $u_i - u_j$ be associated with the O quadric Q_{0ij}, the type $u_i + u_j + u_k$ with the O quadric Q_{ijk}, the transposition (ik) with the involution I_{ik} determined by the points P_{ik}, and the element A_{ijk} with the involution I_{0ijk} $(i, j, k = 1, \cdots, 9)$. Then we find that the involutions I permute the 120 O quadrics just as the generators of $e_{9,2}$ permute the 120 types of invariants. Hence

(38) *The infinite system of irrational invariants conjugate to $u_1 - u_2$ under $e_{9,2}$ divide into 120 conjugate sets such that the infinite number in each set are congruent to each other mod. σ. Thus $e_{9,2}$ has an invariant subgroup $i_{9,2}$ of infinite order whose factor group $f_{9,2}$ has the order $8!8640 = 9!960$ and is isomorphic with the $G_{8,2}$ of § 3 (12).*

There remains to be considered the construction of the invariant subgroup $i_{9,2}$. Since the elements of $i_{9,2}$ leave $u_i - u_j$ invariant mod. σ to within sign they must be either involutory of the form $u_i' = -u_i + m_i\sigma$ or parabolic of the form $u_i' = u_i + l_i\sigma$ $(i = 1, \cdots, 9)$. Here $m_1 + \cdots + m_9 = 2$ and $l_1 + \cdots + l_9 = 0$ because σ is invariant. From (37) we see that the l and m are integers or fractions with denominator 3. Moreover since $u_i - u_j$ is altered by at most an integer multiple of σ, the differences of the l's or of the m's are integers. Hence

(39) *The group $i_{9,2}$ has an invariant abelian subgroup of index two whose elements are of the form $u_i' = u_i + (\lambda_i - r/3)\sigma$; the remaining elements of $i_{9,2}$ all are involutory of the form $u_i' = -u_i + (\mu_i - r/3)\sigma$ $(i = 1, \cdots, 9)$. Here $r = 0, 1, 2$ and λ_i, μ_i are integers such that $\sum_i \lambda_i = 3r$ and $\sum_i \mu_i = 3r + 2$.*

In order to prove that the arithmetical conditions of (39) are sufficient let us develop the subgroup of $e_{n,k}$ which arises from transformations C_m^2 with a symmetrical set of 8 F-points and one isolated F-point say p_9. If these have orders ρ_1 and ρ_2 then $8\rho_1^2 + \rho_2^2 = m^2 - 1$ and $8\rho_1 + \rho_2 = 3(m - 1)$. Eliminating ρ_2 we have $36\rho_1^2 = (m - 1)[24\rho_1 + 1 - 4(m - 1)]$ whence

$m-1=4n$. Then $n=(3\rho_1-4n)^2$ or $n=l^2$ and $3\rho_1=l(4l\pm 1)$. If we set $l=3\nu$ or $l=3\nu\mp 1$ we find two types of transformation, the ambiguous sign being accounted for by a change of sign of ν. These types are

$$D(\nu):\quad \begin{aligned} m &= 36\nu^2+1, \\ \rho_1 &= \nu(12\nu+1), \\ \rho_2 &= 4\nu(3\nu-2); \end{aligned} \qquad C(\nu):\quad \begin{aligned} m &= 4(3\nu+1)^2+1, \\ \rho_1 &= (3\nu+1)(4\nu+1), \\ \rho_2 &= 4(3\nu+1)(\nu+1). \end{aligned}$$

To verify that these transformations exist we note that $C(-1)$ is E_{17} of § 2 which we denote here by E_9 to indicate that p_9 is an ordinary point; and that $C(0)$ denoted hereafter by F_9 is the known transformation of order 5 with a 4-fold point at p_9 and simple points at $p_1, \cdots, p_8$. By direct multiplication we find that $C(\nu)E_9=D(\nu+1)$ and $C(\nu)F_9=D(\nu)$. Hence

$$D(\nu+1)E_9=C(\nu) \quad \text{and} \quad D(\nu)F_9=C(\nu);$$

also $C(\nu)E_9F_9=D(\nu+1)F_9=C(\nu+1)$. Since $C(0)=F_9$ we have $C(\nu)=F_9(E_9F_9)^\nu=(F_9E_9)^\nu F_9$. Finally if we set $D(1)=D_9=F_9E_9$ then $D(\nu)=(F_9E_9)^\nu=D_9^\nu$ and $C(\nu)=F_9D^{-\nu}=D_9^\nu F_9$. Hence

(40) *The types of C_m^2 with 8 symmetrical and one isolated F-point lead to elements of $e_{9,2}$ which lie in the invariant subgroup $i_{9,2}$ and constitute a dihedral subgroup of infinite order generated by the involutions F_9 and E_9 whose product D_9 is parabolic. In the dihedral group the generators belong to distinct conjugate sets.*

The last statement follows from the fact that D_9 transforms $C(\nu)$ into $C(\nu-2)$. That all these elements lie in $i_{9,2}$ follows from the parametric expressions of F_9 and E_9 which from (37) are

$$F_9:\ \begin{aligned} u_1' &= -u_1+\sigma/3, \\ &\cdots\cdots \\ u_8' &= -u_8+\sigma/3, \\ u_9' &= -u_9+\sigma/3-\sigma; \end{aligned} \qquad E_9:\ \begin{aligned} u_1' &= -u_1, \\ &\cdots\cdots \\ u_8' &= -u_8, \\ u_9' &= -u_9+2\sigma; \end{aligned} \qquad D_9:\ \begin{aligned} u_1' &= u_1-\sigma/3, \\ &\cdots\cdots \\ u_8' &= u_8-\sigma/3, \\ u_9' &= u_9-\sigma/3+3\sigma. \end{aligned}$$

We see that in D_9 the differences of the integers λ of (39) all are divisible by 3 and that this is true as well for any product $D_1^{\nu_1}D_2^{\nu_2}\cdots D_9^{\nu_9}$ where D_i is formed for the point p_i as D_9 for p_9. Hence we cannot hope to get in this way all possible elements of the form (39). However the product

$$C_{2,1}=E_2E_1:\ \begin{aligned} u_1' &= u_1+2\sigma, \\ u_2' &= u_2-2\sigma, \\ u_3' &= u_3, \\ &\cdots\cdots \\ u_9' &= u_9 \end{aligned}$$

is not subject to this defect. Then the product

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{2,9}^{\rho_9} \tag{41}$$

will be the general element of the abelian subgroup of (39) if

$$\begin{aligned}
-(\nu_2 + \cdots + \nu_9)/3 + 2(\rho_2 + \cdots + \rho_9) &= \lambda_1 - r/3, \\
3\nu_2 - (\nu_2 + \cdots + \nu_9)/3 - 2\rho_2 &= \lambda_2 - r/3, \\
\cdots\cdots\cdots\cdots \\
3\nu_9 - (\nu_2 + \cdots + \nu_9)/3 - 2\rho_9 &= \lambda_9 - r/3.
\end{aligned}$$

These 9 equations are linearly related so that we need to satisfy only the last 8. Let $\lambda_j - \lambda_1 = 3a_j - 2b_j$ $(j = 2, \cdots, 9)$ where $b_j = 0, 1, 2$. Then $2\sum_j b_j = 3\sum_j a_j + 9\lambda_1 - 3r$ or $\sum_j b_j = 3s$, where s is an integer. If then we set $\rho_2, \cdots, \rho_9$ equal to $b_2, \cdots, b_9$ and $\nu_2 + \cdots + \nu_9$ equal to $a_2 + 2s$, $\cdots$, $a_9 + 2s$ respectively the equations are satisfied for all permissible integer values of λ and r. Hence

(42) *Every element of the form described in* (39) *actually occurs in* $i_{9,2}$. *Every element of the abelian subgroup can be expressed as a product* (41) *in one and only one way if we restrict the exponents* $\rho_2, \cdots, \rho_9$ *to be* $0, 1, 2$. *The remaining elements of* $i_{9,2}$ *can be expressed as a product of* E_1 *and* (41).

A consequence of this theorem is that the infinite number of types of ternary Cremona transformations with 9 F-points can be arranged in $960 \cdot 3^8 \cdot 2$ classes such that each class contains an infinite number of types depending on the unrestricted variation of the 8 integers $\nu_2, \cdots, \nu_9$.

We now turn to the set P_8^3 on E^4 with parameters $u_1, \cdots, u_8$. The group, $e_{8,3}$ is in most respects analogous to $e_{9,2}$ but we shall find some points of difference. The set of irrational invariants of $e_{8,3}$ conjugate to $u_1 - u_2$ is infinite in number. If however P_8^3 is self associated, i. e., if $\sigma = 0$, they reduce to a set of 63 which comprise 28 of the form $u_i - u_j$ and 35 of the form $u_i + u_j + u_k + u_l$. If in the finite geometry of the theta functions for $p = 3$ we identify these invariants with the 63 points P_{ij} and P_{ijkl} and if we identify the generators (ik) and A_{ijkl} of $e_{8,3}$ with the involutions on the 63 points, it is easy to verify that the invariants are permuted by the generators just as the points are permuted by the involutions. Hence

(43) *The infinite set of irrational invariants conjugate to* $u_1 - u_2$ *under* $e_{8,3}$ *divide into* 63 *conjugate sets such that the infinite number in each set are congruent mod.* σ. *The* $e_{8,3}$ *has an invariant subgroup* $i_{8,3}$ *of infinite order whose factor group* $f_{8,3}$ *is of order* $7!288$ *and simply isomorphic with* $G_{7,3} = G_{7,2}$.

To determine the construction of $i_{8,3}$ we note as before that its elements must be either parabolic of the form $u_i' = u_i + (\lambda_i - r/2)\sigma$ or involutory of the form $u_i' = -u_i + (\mu_i - r/2)\sigma$ $(i = 1, \cdots, 8)$, where λ_i and μ_i are

integers, r is $0, 1$, and $\sum_i \lambda_i = 4r$, $\sum_i \mu_i = 4r + 2$. In order to prove that these are the only conditions on the coefficients we use (25) § 5 to determine all types of regular Cremona transformations in S_3 with 8 F-points 7 of which form a symmetrical set. The algebra runs along about as in the preceding case. The resulting types, as elements of $g_{8,\,3}$ in the notation at the beginning of § 5 have the following matrices:

$$C(\nu):\quad \begin{array}{c|l|l|l|} & & \quad 1 & \quad 7 \\ \hline & 14(2\nu+1)^2+1 & -7\nu(2\nu+1) & -(2\nu+1)(7\nu+4) \\ \hline 1 & 14\nu(2\nu+1) & -(7\nu^2-1) & -\nu(7\nu+4) \\ \hline 7 & 2(2\nu+1)(7\nu+4) & -\nu(7\nu+4) & -(7\nu^2+8\nu+3),\ -(7\nu^2+8\nu+2) \\ \hline \end{array}\ ;$$

$$D(\nu):\quad \begin{array}{c|c|c|c|} & & 1 & 7 \\ \hline & 56\nu^2+1 & -7\nu(2\nu+1) & -\nu(14\nu-1) \\ \hline 1 & 14\nu(2\nu-1) & -(7\nu^2-1) & -\nu(7\nu-4) \\ \hline 7 & 2\nu(14\nu-1) & -\nu(7\nu+4) & -(7\nu^2-1),\ -7\nu^2 \\ \hline \end{array}\ .$$

That these types exist for all integer values of ν can be shown as follows. The type $C(0)$ is C_{15} of § 5 and $C^2(0) = 1$. The type $C(-1)$ also of order 15 and period two can be reduced to the known type C_5 of § 5 by multiplying it by a product of elements A. We find that $C(-1)C(0) = D(1)$ and that $D(\nu)D(1) = D(1)D(\nu) = D(\nu+1) = [D(1)]^{\nu+1}$. Thus $C(0)$ and $C(-1)$ generate a dihedral group in $g_{8,\,3}$ whose cyclic subgroup is generated by $D(1)$. If we denote $D(1)$ by D_1 and $C(0)$ by C_1 to indicate that the point p_1 is isolated we find from (37) that the corresponding elements of $e_{8,\,3}$ are

$$C_1:\quad \begin{aligned} u_1' &= -u_1 + 2\sigma, \\ u_2' &= -u_2, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u_8' &= -u_8; \end{aligned} \qquad\qquad D_1:\quad \begin{aligned} u_1' &= u_1 - \sigma/2 + 4\sigma, \\ u_2' &= u_2 - \sigma/2, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u_8' &= u_8 - \sigma/2; \end{aligned}$$

$$C_{2,\,1} = C_2\, C_1:\quad \begin{aligned} u_1' &= u_1 + 2\sigma, \\ u_2' &= u_2 - 2\sigma, \\ u_3' &= u_3, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u_8' &= u_8. \end{aligned}$$

In all these elements the differences of the λ's and μ's are even so that some missing elements are yet to be found. The involutions A_{1234} and A_{5678} of

$e_{8,\,3}$ are the same mod. σ. Hence their product

$$u_i' = u_i + \sigma/2, \qquad u_j' = u_j + \sigma/2 - \sigma \qquad (i = 1, \cdots, 4;\ j = 5, \cdots, 8)$$

is found in $i_{8,\,3}$. It does not have the defect noted above. By combin'ng a properly selected pair of such products we find the simple element of $i_{8,\,3}$:

$$E_{2,\,1}\colon\ u_1' = u_1 + \sigma, \qquad u_2' = u_2 - \sigma, \qquad u_3' = u_3, \qquad \cdots, \qquad u_8' = u_8.$$

We can now prove as before the theorem:

(44) *The invariant subgroup* $i_{8,\,3}$ *of* $e_{8,\,3}$ *contains an invariant abelian subgroup of index two whose elements all are parabolic of the form* $u_i' = u_i + (\lambda_i - r/2)\sigma$; *the remaining elements of* $i_{8,\,3}$ *all are involutory of the form*

$$u_i' = -u_i + (\mu_i - r/2)\sigma$$

where λ_i, μ_i, r *are any integers such that* $\sum_i \lambda_i = 4r$, $\sum_i \mu_i = 4r + 2$, $r = 0, 1$ $(i = 1, \cdots, 8)$. *Every element of the abelian subgroup can be expressed in a single way by a product*

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_8^{\nu_8} C_{3,1}^{\nu_{31}} C_{4,1}^{\nu_{41}} \cdots C_{8,1}^{\nu_{81}} E_{3,1}^{\rho_{31}} E_{4,1}^{\rho_{41}} \cdots E_{8,1}^{\rho_{81}}$$

such that $\nu_{j,\,1}$, $\rho_{j,\,1} = 0, 1$ $(j = 3, \cdots, 8)$.

In fact if $\lambda_j - \lambda_2 = 4a_{j,\,2} - 2b_{j,\,2} - c_{j,\,2}$ where $b_{j,\,2}, c_{j,\,2} = 0, 1$ then the exponents in the product are precisely $\rho_{j1} = c_{j,\,2}$, $\nu_{j,\,1} = b_{j,\,2}$, $\nu_2 = k$, $\nu_j = k + a_{j,\,2}$ where $k = \sum_j a_{j,\,2} + 2\lambda_2 - r$.

By means of the foregoing theorems properties of the groups $G_{8,\,3}$ and $G_{9,\,2}$ become accessible and further theorems in regard to particular point sets can be established. Thus we see at once from (43) and (44) that

(45) *If* P_8^3 *is the set of base points of a net of quadrics there are only* 36 *projectively distinct sets congruent in some order to* P_8^3.

This could be inferred from the theory of the theta functions for $p = 3$. The following fact is more novel:

(46) *If* P_8^3 *is a set of* 8 *nodes of a quartic surface but not the set of base points of a net of quadrics* then there are only* $2^6 \cdot 36$ *projectively distinct sets congruent in some order to* P_8^3.

Indeed the parameters on E^4 of the 36 sets of (45) are of the type $u_1, \cdots, u_8$ or of the type $u_i - (u_1 + \cdots + u_4)/2$, $u_j + (u_1 + \cdots + u_4)/2$ $(i = 1, \cdots, 4;$ $j = 5, \cdots, 8)$. If in each of these 36 sets $\sigma = \omega/2$ be added to any two or any four of the parameters all $36 \cdot 2^6$ sets of (46) are obtained. The group $e_{8,\,3}$ contains an invariant subgroup $i_{8,\,3}'$ which leaves each of the $36 \cdot 2^6$ sets unaltered. The factor group $f_{8,\,3}'$ has the order $8!36 \cdot 2^6$ and is isomorphic

* Cayley's *dianome* surface, *Collected Mathematical Papers*, vol. 7, p. 133.

with that group of operations on the theta functions ($p = 3$) which is generated by the transformation of the periods and by the addition of half periods to the argument.

Similar theorems can be stated for P_9^2.

(47) *A pencil of plane cubic curves can be transformed by ternary Cremona transformation into only* 960 *projectively distinct pencils of cubics. A pencil of plane sextics with* 9 *double points can be transformed into only* $2^8.960$ *projectively distinct pencils of sextics.*

Of course these theorems are only special cases of the more general theorem that if $\sigma = \omega/r$ (r an integer) the number of projectively distinct sets congruent in some order to P_8^3 or P_9^2 is finite and equal respectively to $36N$ or $960N$, where N is the order of the finite group derived from $i_{8,3}$ or $i_{9,2}$ by the assumption $r\sigma \equiv 0$.

The remarkable utility of $e_{n,k}$ in the discussion of the special cases of $G_{n,k}$ noted above seems not to extend to other cases. The other cases however are distinguished sharply from the ones here treated by the fact that the invariant quadratic form (35) does not admit the transformation $u_i' \equiv u_i + \omega/(k+1)$.

7. The isomorphisms of $G_{n,k}$

In this paragraph some isolated facts previously noted are brought together and some of the properties of $G_{n,k}$ are summarized.

(48) *If* $G_{n,k}$ *is infinite it is simply isomorphic with* $g_{n,k}$ *and* $e_{n,k}$. *The* $G_{n,k}$ *contains an invariant subgroup* $\Gamma_{n,k}$ *of index two whose elements are products of an even number of generators. The* $G_{9,2} = G_{9,5}$ *and the* $G_{8,3}$ *have the constitution described in* (38), (39) *and* (43), (44) *of* § 6. *The invariant subgroup corresponding to* $i_{9,2}$ *is contained in* $\Gamma_{9,2}$; *that corresponding to* $i_{8,3}$ *is not completely contained in* $\Gamma_{8,3}$.

The simple isomorphism follows from the statement preceding (18) of § 4 and from (33) of § 6. The existence of $\Gamma_{n,k}$ then follows from (31) of § 5 or from (33). The element C_1 of $i_{8,3}$ has the determinant -1 while both E_9 and F_9 of $i_{9,2}$ have the determinant 1.

(49) *The* $G_{8,2} = G_{8,4}$ *is in* 1 *to* 2 *isomorphism with* $g_{8,3}$ *and* $e_{8,2}$. *It contains an invariant subgroup* $\Gamma_{8,2} = \Gamma_{8,4}$ *which is simple.*

For according to (26) § 5, $g_{8,2}$ contains an invariant involution I and this, the E_{17} of § 2, leads to the identity in $G_{8,2}$. The corresponding element of $e_{8,2}$ is $u_i' = -u_i$ ($i = 1, \cdots, 8$) of determinant 1 whence the factor group $G_{8,2}$ still has an invariant subgroup of index two. That this group is simple is proven in § 209 of Dickson's *Linear Groups*.

(50) *The* $G_{7,2} = G_{7,3}$ *is in* 1 *to* 2 *isomorphism with* $g_{7,2}$ *and* $e_{7,2}$ *and itself is a simple group.*

Here $g_{7,2}$ and $e_{7,2}$ have the same invariant involution described under (49) but in this case it has a determinant -1 so that the factor group is $\Gamma_{7,2} = G_{7,2}$. In § 3 we have seen that $G_{7,2}$ is isomorphic with the double tangent group which is known to be simple.

(51) *The $G_{6,2}$ is simply isomorphic with $g_{6,2}$ and $e_{6,2}$. The invariant subgroup $\Gamma_{6,2}$ is the known simple group of order* 25920.

(52) *The $G_{k+3,k} = G_{k+3,1} = G_{(k+3)!}$ is in 1 to 2^{k+2} isomorphism with $g_{k+3,k}$ and $e_{k+3,k}$. It has an invariant subgroup $\Gamma_{k+3,k}$ of index two which is simple when $k > 1$.*

We naturally assume that $k > 1$ to exclude the case of a P_4^1 or Q_4^1. Then $g_{k+3,k}$ contains an invariant abelian subgroup of order 2^{k+2} generated by elements which are products like that of $A_{1,\dots,k+1}$ with the transposition $(k+2, k+3)$. All these according to § 2 lead to sets P_{k+3}^k which are projectively equivalent in the identical order whence they correspond to the identity in $G_{k+3,k}$. Also they are of determinant 1 so that $G_{k+3,k}$ still has its subgroup $\Gamma_{k+3,k}$. Since the transpositions have a determinant -1, $\Gamma_{k+3,k}$ is isomorphic with the simple alternating group $g_{\frac{1}{2}(k+3)!}$.

The above summary includes all cases in which the Cremona group $G_{n,k}$ in $\Sigma_{k(n-k-2)}$ exists though $g_{n,k}$ and $e_{n,k}$ persist for values $n = k+2, k+1$. Let us state one further theorem:

(53) *When $n = 2\nu$ or $n = 2\nu + 1$ the group $G_{n,k}$ contains ν distinct conjugate sets of involutions represented by the ν distinct types of involutions in $G_{n!}$.*

That the sets are distinct follows from the fact that in the collineation groups $e_{n,k}$ and $g_{n,k}$ they are of projectively different kinds. In case $G_{n,k}$ is finite it contains no other involutions since there are no other kinds in the finite geometry. In the infinite cases $G_{9,2}$ and $G_{8,3}$ however there exist the two new conjugate sets described in § 6, (39) and (44). The sets are distinguished by the conditions $r = 0$, $r \neq 0$. These new sets serve to generate the invariant subgroups $i_{9,2}$ and $i_{8,3}$.

Possibly the most striking feature of the above résumé is the paucity of facts concerning the general $G_{n,k}$.

8. Invariants of $G_{n,k}$

Apparently nothing can be said in a general way of the invariants of the extended group $G_{n,k}$. Each case presents features peculiar to itself and must be separately treated. The case of $G_{6,2}$ for which a complete system can be obtained is reserved for Part III of this account. For the case $G_{7,2}$ a rather interesting linear system of irrational invariants is given below from which rational invariants of $G_{7,2}$, i. e., of the ternary quartic, can be calculated. Some general conclusions as to the existence of certain invariants for the infinite cases $G_{9,2}$ and $G_{8,2}$ also are given.

Let us use for P_7 with points $p_1, \cdots, p_7$ the basis notation of § 3 according to which the degenerate cubics of the net on P_7^2 are named as follows: Q_{ij} is the line on p_i, p_j and the conic on the other five points; Q_{0i} is the point p_i and the cubic with node at p_i $(i, j = 1, \cdots, 7)$. The net of cubics on P_7^2 maps S_2 on a plane S_2' in such a way that the 28 degenerate cubics map into the 28 double tangents of a general quartic curve C^4 in S_2'. This C^4 acquires a node when either two points p coincide in some direction, or three points lie on a line, or six points lie on a conic. Hence the discriminant Δ of C^4 breaks up into 63 irrational factors which we shall denote as follows: $\delta_{ik} = 0$ is the condition that p_i and p_k coincide; $\delta_{0i} = 0$ is the condition that the six points other than p_i are on a conic; and $\delta_{0ijk} = 0$ is the condition that the points p_i, p_j, p_k are on a line.* Thus the double tangents and the discriminant factors are permuted under the operations of $G_{7,\,2}$ as the O quadrics and the points of S_5 are permuted under the group of the null system.

The discriminant factors do not behave alike with reference to the set P_7^2. The factor δ_{0i} is of degree two in the coördinates of each point other than p_i. The factor δ_{0ijk} is of degree one in the coördinates of p_i, p_j, p_k. But the factor δ_{ij} which vanishes if p_i and p_j coincide in some direction cannot be expressed by a single condition on the coördinates of P_7^2. When these points coincide both δ_{0ijk} and δ_{0k} must vanish. Hence the occurrence of a factor δ_{ij} in any product of discriminant factors must be indicated by the degree to which other factors vanish when $p_i = p_j$. The discriminant itself must be unaltered by permutation of the points whence it is to within a numerical factor the product of the seven squared factors δ_{0i}^2 and of the thirty-five squared factors δ_{0ijk}^2. Hence Δ is of degree 54 in the coördinates of each point and has a zero of order $20 = 2 + 18$ for each coincidence (i, j) rather than a zero of order 2 corresponding to the squared factor δ_{ij}^2. Since Δ is an invariant of C^4 of degree 27 in the coefficients the first invariant of C^4 of degree 3 in the coefficients should be of degree 6 in the coördinates of each point of P_7^2 and should have a zero of order 2 for each coincidence (i, j). This invariant is given below in (69).

It is customarily true that rational invariants can be expressed as symmetric functions of irrational invariants and the question at once arises as to whether there are irrational invariants of P_7^2 of degree 3 in the coördinates of each point which vanish at least once for each coincidence (ij). If we examine the product

$$(514)(624)(235)(136)(127)(347)(567), \tag{54}$$

where (ijk) is the ternary determinant symbol $(p_i\,p_j\,p_k)$, we see that it is homogeneous and of degree 3 in the coördinates of each point and that it

* Cf. De Paolis, Atti di Lincei, ser. 3, vol. 1 (1877), p. 511, and vol. 2 (1878), p. 851; see also Snyder (loc. cit.).

vanishes once and only once for each coincidence. It corresponds therefore to the following product of seven factors of Δ

$$\delta_{0514}\,\delta_{0624}\,\delta_{0235}\,\delta_{0136}\,\delta_{0127}\,\delta_{0347}\,\delta_{0567}\,. \tag{55}$$

According to F. G., II (3), (4), the corresponding seven points in the finite geometry lie in a plane and any two are syzygetic, i. e., the seven points are in a Göpel plane. According to F. G., I (19), there are 135 such Göpel planes. The combination (54) is well known in other connections.* It is unaltered to within sign by 168 permutations of the points so that it is one of a set of 30 similar products. If we carry out on (55) the Cremona involution A_{123} the factors δ_{0514}, δ_{0624}, δ_{0347} are unaltered; the factors δ_{0235}, δ_{0136}, δ_{0127} become respectively δ_{15}, δ_{26}, δ_{37}; while the factor δ_{0567} becomes δ_{04}. Thus (54) is transformed into

$$(514)\,(624)\,(347)\,\Delta_4, \tag{56}$$

where $\Delta_4 = 0$ is the condition expressed in terms of the points that the points other than p_4 are on a conic. Of this product it is true also that it is homogeneous of degree 3 in the coördinates of each point, vanishes at least once for each coincidence, and vanishes twice for the coincidences $(5, 1)$, $(6, 2)$, $(3, 7)$. The corresponding product of factors of Δ is therefore

$$\delta_{0514}\,\delta_{0634}\,\delta_{0347}\,\delta_{04}\,\delta_{51}\,\delta_{62}\,\delta_{37}\,. \tag{57}$$

These factors again correspond to the seven points of a Göpel plane. There are 7.15 possible expressions like (56) so that the products corresponding to the 135 Göpel planes all are included in the types (54) and (56). Hence

(58) *The* 63 *irrational factors of the discrimination of* C^4 *can be grouped in* 135 *ways into products of the seven factors which correspond to the points of a Göpel plane. These products form a conjugate set of irrational invariants of* C^4 *which are permuted to within sign or to within a common outstanding factor by the operations of* $G_{7,\,2}$.

If in each of these *Göpel invariants* we regard one of the points say p_7 as a variable point we obtain 135 cubics on the remaining points P_6^2. An inspection of (54) and (56) shows that the cubics are those which in I, § 4, were found to be the maps of the 45 tritangent planes of the cubic surface C^3 mapped from P_6^2. It is therefore possible to express the 135 Göpel invariants in terms of $\bar{a}, \cdots, \bar{f}$ and $a, \cdots, f$ of I, § 4. This is accomplished by noting that from I (46), (47) we have

$$\begin{aligned} (531)(461)(342)(562) + (532)(462)(341)(561) &= -\overline{cf}, \\ (531)(461)(342)(562) - (532)(462)(341)(561) &= -d_2, \end{aligned} \tag{59}$$

* Cf. Weber, *Algebra*, II, p. 540 (3); see also H. S. White, Proceedings of the National Academy of Sciences, vol. I, p. 464 (1915).

where $\overline{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \overline{cf})$, $a_2 = \sum \bar{a}\bar{b}$, and $d_2 = \Delta_7 = 0$ is the condition that P_6^2 be on a conic. Moreover from I (29),

$$(c + f) = 4(547)(217)(367).$$

Hence by using (59) we find that

$$\begin{aligned} &8(531)(461)(342)(562)(547)(217)(367) \\ &\qquad\qquad = -(\overline{cf} + d_2)(c + f) \equiv [cf +], \\ (60)\quad &8(523)(462)(341)(561)(547)(217)(367) \\ &\qquad\qquad = (\overline{cf} - d_2)(c + f) \equiv [cf -], \\ &8\Delta_7(547)(217)(367) = 2d_2(c + f) \equiv [cf]. \end{aligned}$$

By using the parallel permutations (12), (23456); $(ad)(be)(cf)$, $(adbfe)$ of I (28), an odd permutation being accompanied by a change of sign of d_2 we get from (60) the 45 Göpel invariants which contain as factors the 15 tritangent planes of the form $c + f$. There remain to be determined the expressions for the 90 Göpel invariants of type (56) with factor other than Δ_7. In the first of equations I (38) we find

$$4\Delta_2(127) = (531)(461)(a + d) - (341)(561)(b + e).$$

If we multiply by $2(245)(236)$ so as to obtain the same degree in all the points, and if then we replace the coefficients of $(a + d)$ and $(b + e)$ by means of relations derived from (59) by permutation we get

$$\begin{aligned} (61)\quad 8\Delta_2(127)(245)(236) &= -(\overline{ad} - d_2)(a + d) \\ &\qquad + (\overline{be} + d_2)(b + e) \equiv [ad, be], \end{aligned}$$

which is the left member of the last equation of I (43). Had we used the factors $2(234)(256)$ or $2(235)(246)$ the remaining members of this set of equations would have obtained. Hence

(62) *The* 135 *Göpel invariants for isolated* p_7 *divide into* 45 *sets of three such that the three in a set contain as factor a tritangent plane of the cubic surface* C^3 *of* P_6^2. *These sets divide into* 15 *of the kind,* $[cf +]$, $[cf -]$, $[cf]$; *and* 30 *of the kind* $[ad, be]$, $[be, cf]$, $[cf, ad]$. *All the sets are obtained from the two given sets by transposing two letters and changing the sign of* d_2.

We can use (61) and (60) to define the sign of Δ_i in terms of the sign of d_2 as given by (59).

It is clear from (60) that the first set of three in (62) are linearly related. These three Göpel invariants correspond in the finite geometry to three Göpel planes with a common null line. According to F. G., I (19), (22), there are 315 null lines all of which are conjugate and each of which is on three Göpel planes. There must be therefore 315 such relations as follows:

$$
(63)\qquad
\begin{aligned}
&1^\circ \quad [cf] + [cf+] + [cf-] = 0,\\
&2^\circ \quad [ad, be] + [be, cf] + [cf, ad] = 0,\\
&3^\circ \quad [cf] + [be] + [ad] = 0,\\
&4^\circ \quad [ad, be] + [ad-] + [be+] = 0,\\
&5^\circ \quad [ad, be] + [be, ad] + [cf] = 0,\\
&6^\circ \quad [ab, de] + [bc, ef] + [ca, fd] = 0.
\end{aligned}
$$

The number of relations of types $1^\circ, \cdots, 6^\circ$ is respectively 15, 30, 15, 90, 45, 120. All of these relations are mere identities in the letters or they are satisfied due to $a + b + \cdots + f = 0$ except those of type 6°. They exist by virtue of the identity

$$
(64)\qquad
\begin{aligned}
&\overline{bc}(b+c) + \overline{ca}(c+a) + \overline{ab}(a+b)\\
&\qquad = \overline{ef}(e+f) + \overline{fd}(f+d) + \overline{de}(d+e),
\end{aligned}
$$

which is proved below. Hence

(65) *Corresponding to the fact that the* 135 *Göpel planes pass by threes through* 315 *null lines, the* 135 *Gopel invariants are connected by the* 315 *three-termed linear relations of* (63).

We see that the 15 invariants $[cf]$ can be expressed in terms of 5. For $[ab] + [ac] + \cdots + [af] = 8d_2 a$ whence $d_2 a, \cdots, d_2 f$ subject to $\sum_6 d^2 a = 0$* will serve to express all of type $[cf]$. Furthermore by adding terms like $d_2 a$ to $[cf+]$ and $[cf-]$ these invariants are reduced to expressions of the form $\overline{cf}(c+f)$. According to (63) 4° invariants of type $[ad, be]$ can be expressed in terms of types $[cf+]$ and $[cf-]$. Thus all the Göpel invariants can be expressed by means of terms like $d_2 a$ and $\overline{cf}(c+f)$. Terms of the latter kind are themselves linearly related as in (64). But the ten relations (64) are themselves a consequence of five linear relations. For if we symmetrize with the help of $\bar{a}$ and a we find that

$$
(66)\qquad \overline{ab} + \overline{ac} + \cdots + \overline{af} = 6\bar{a}^2 + a_2,
$$

$$
(67)\qquad \overline{ab}(a+b) + \overline{ac}(a+c) + \cdots + \overline{af}(a+f) = 2\sum_6 \bar{a}^2 a.
$$

The right member of (67) is symmetric. If then we subtract the sum of relations (67) formed for d, e, f from the sum formed for a, b, c the relation (54) is obtained. Thus by equating the left members of the 6 relations (67) five independent linear relations on the 15 terms $\overline{cf}(c+f)$ are obtained and only 10 of these terms are independent. Hence

* Here and hereafter the number under Σ indicates the number of terms of the type following which are to be used in symmetrizing for six letters $a, \cdots, f$ or $\bar{a}, \cdots, \bar{f}$.

(68) The 135 *Göpel invariants lie in a linear system of irrational invariants of which only* 15 *are linearly independent. They give rise in* Σ_6 *to a linear system of spreads of order seven invariant under* $G_{7,\,2}$.
For if we put P_7^2 in the canonical form of I, p. 53, it is easy to see from (54) and (56) that u^2 factors out of each Göpel invariant. According to I (68) the simplest invariant linear system in Σ_6 under $G_{7!}$ has the order 8. Thus the members of this system which contain an additional factor u constitute the simplest linear system in Σ_6 invariant under the extended group $G_{7,\,2}$.

Any symmetric function of the squares of the 135 Göpel invariants or any unsymmetric functions of these squares whose terms are permuted by the operations of $G_{7,\,2}$ is a rational invariant of $G_{7,\,2}$ and therefore also of the allied ternary quartic C^4. The invariant of lowest degree is obtained from the sum of the squares. It is

$$I_1 = \sum_{90} [ad, be]^2 + \sum_{15} [ad+]^2 + \sum_{15} [ad-]^2 + \sum_{15} [ad]^2.$$

By making use of (63) 4° this can be written

$$I_1 = 2\sum_{90} [ad-][be+] + 7\sum_{15} [ad+]^2 + 7\sum_{15} [ad-]^2 + \sum_{15} [ad]^2.$$

This reduces eventually to the following explicit expression in a and $\bar{a}$:

$$(69)\qquad \begin{aligned} I_1 = 12\,[\sum_{6} a^2 (8d_2^2 + 27a_3\,\bar{a} - 24a_2\,\bar{a}^2 - 22\bar{a}^4) \\ + 4\sum_{15} ab\,\bar{a}\bar{b}\{4(\bar{a}^2+\bar{b}^2) + 5\bar{a}\bar{b}\}]. \end{aligned}$$

It is for the surface C^3 a covariant quadric of degree six.

With the Göpel invariants as elements it is possible to form other conjugate sets of irrational invariants which correspond in the finite geometry to other configurations such as the O and E quadrics. Enough has been said above to show that a theory of the invariants of the quartic based on the invariants of $G_{7,\,2}$ may be feasible. The formulæ developed above will be useful for the problem considered in Part III of determining covariants of C^3, particularly the linear covariants.

The following remarks on the invariants of $G_{9,\,2}$ may serve to illustrate also the infinite case $G_{8,\,3}$. The set P_2^9 is on a cubic curve E^3 whose coefficients are of degree 3 in the coördinates of each point. The invariants S and T of E^3 are invariants of the extended group $G_{9,\,2}$. If we take P_9^2 in the canonical form of I, § 53, so that the coördinates of $p_5, p_6, \cdots, p_9$ are $y_1^{(i)}$, $y_2^{(i)}$, u $(i = 1, \cdots, 5)$ then P_9^2 is represented by the point y, u of Σ_{10} and the invariants S and T give rise to invariant spreads in Σ_{10} of orders $48 - \lambda$ and $72 - \mu$. Here λ, μ are the number of additional factors u which occur in the S and T of E^3 formed for the canonical form of P_9^2—numbers which could

be determined by a somewhat tedious calculation. Hence in Σ_{10} $G_{9,2}$ has the pencil of invariant spreads $S^3 + kT^2 = 0$. The base of this pencil is a manifold M_8 determined by $S = T = 0$, which breaks up into two parts M'_8 and M''_8. The one part M'_8 is the map of sets P_9^2 for which E^3 has a cusp. The other part M''_8 is the map of sets P_9^2 for which E^3 vanishes, i. e., sets which are the base points of a pencil of E^3's. Both of these *manifolds* M'_8 *and* M''_8 *are rational.* For in the case of M'_8 the P_9^2 can be put in the form $x_{i1} = t_i^3$, $x_{i,2} = t_i$, $x_{i,3} = 1$ $(i = 1, \cdots, 9)$. The transformation $t' = \alpha t$ $(\alpha \neq 0)$ gives rise to a projectively equivalent P_9^2 so that only the ratios of the 9 parameters are essential. If the above set P_9^2 be transformed linearly into the canonical form, the coördinates y, u appear as functions of the 9 homogeneous parameters t. In the case of M''_8 the same result follows since the coördinates of the 9th base point of a pencil of cubics are rational functions of the coördinates of the other 8 base points. Though both of the manifolds M'_8 and M''_8 are invariant under $G_{9,2}$ their behavior is quite different. *On the invariant manifold* M''_8 *the infinite group* $G_{9,2}$ *effects only a finite number of distinct transformations* which constitute a group of order $8!8640$. In fact this group is merely a representation of the factor group of the invariant subgroup $i_{9,2}$ of $G_{9,2}$ described in § 6. *On the invariant manifold* M'_8 *the group* $G_{9,2}$ *effects transformations which are represented on the parameters* $t_1, \cdots, t_9$ *by the operations of* $e_{9,2}$ *of* § 6. To prove this one has only to test the effect on the parameters t of the transformation A_{123} and note that it is the same as (32) of § 6.

The continuous pencil of invariant spreads $S^3 + kT^2 = 0$ is cut by an infinite but discontinuous system of invariant algebraic spreads $M_9(r)$, r any positive integer. The spread $M_9(r)$ is the map of sets P_9^2 for which the sum of the elliptic parameters on E^3 is a primitive rth period. The degree of this invariant $M_9(r)$ in the coördinates of each point of P_9^2 is 3ν where ν is the number of primitive rth periods. For if $p_1, \cdots, p_8$ be given each member of the pencil of cubics on P_8^2 contains ν points each of which makes up with P_8^2 a P_9^2 of the required type. The ν points run over a curve $C(r)$ which must be of order $3k$ with k-fold points at $p_1, \cdots, p_8$ since its characteristic property is invariant under $G_{8,2}$. Hence $C(r)$ meets an E^3 of the pencil in $9k - 8k = \nu$ points apart from P_8^2. The order of $M_9(r)$ in Σ_{10} is $15\nu - \lambda$ if u^λ factors out of $M_9(r)$ for the canonical form of P_9^2. On each of these invariant spreads $G_{9,2}$ effects only a finite number of distinct transformations—a number which could readily be determined for any particular value of r.

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